

Name: **Solutions**

(100 points total)

1. (40 points) Determine whether the following series converge or diverge. **If a series converges, find its sum.** Justify your answers and state all convergence tests used.

(a)
$$\sum_{n=1}^{\infty} (-1)^n \sqrt{\frac{1}{4} + \frac{1}{n^2}}.$$

Solution. The series diverges by the n -th term test for divergence, since the sequence of terms does not converge to 0, i.e., $\lim_{n \rightarrow \infty} \left((-1)^n \sqrt{\frac{1}{4} + \frac{1}{n^2}} \right) \neq 0$. (In fact, $\lim_{n \rightarrow \infty} \left((-1)^n \sqrt{\frac{1}{4} + \frac{1}{n^2}} \right)$ does not exist.)

(b)
$$\sum_{k=1}^{\infty} a_k, \text{ if the partial sum } s_n = \sum_{k=1}^n a_k \text{ equals } \frac{4n^2}{n^2 + 3}.$$

Solution. The series converges to 4, since the sequence of partial sums converges to 4.

(c)
$$\sum_{n=0}^{\infty} \frac{3^n}{n!}$$

Solution. One could use the ratio test to prove that this series converges. However, there is a simpler approach. We saw that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x . This means that the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x for all x . Thus $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges to e^3 .

$$(d) \sum_{n=2}^{\infty} (-1)^n \frac{5}{3^{2n}}$$

Solution.

$$\sum_{n=2}^{\infty} (-1)^n \frac{5}{3^{2n}} = \sum_{n=2}^{\infty} (-1)^n \frac{5}{(3^2)^n} = \sum_{n=2}^{\infty} 5 \frac{(-1)^n}{(9)^n} = \sum_{n=2}^{\infty} 5 \left(\frac{-1}{9} \right)^n$$

is a geometric series with ratio $r = -1/9$ and first term $a = 5(-1/9)^2 = 5/81$. Since $|r| < 1$, the series converges to

$$\frac{a}{1-r} = \frac{5/81}{1 - (-1/9)} = \frac{1}{18}.$$

2. (15 points) Does $\sum_{n=3}^{\infty} (-1)^n \left(\frac{1 + \cos n}{n^2} \right)$ converge absolutely, converge conditionally, or diverge? State all convergence tests used.

Solution. We first test for absolute convergence:

$$\sum_{n=3}^{\infty} \left| (-1)^n \left(\frac{1 + \cos n}{n^2} \right) \right| = \sum_{n=3}^{\infty} \left(\frac{1 + \cos n}{n^2} \right)$$

(where we are using that fact that $\frac{1 + \cos n}{n^2} \geq 0$ for all $n \geq 3$). For all $n \geq 3$,

$$0 \leq \frac{1 + \cos n}{n^2} \leq \frac{2}{n^2},$$

and $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges since it is a p -series with $p = 2$. Therefore $\sum_{n=3}^{\infty} \left(\frac{1 + \cos n}{n^2} \right)$ converges by the Direct Comparison Test. Hence $\sum_{n=3}^{\infty} (-1)^n \left(\frac{1 + \cos n}{n^2} \right)$ converges absolutely.

3. (30 points) Find the radius **and** interval of convergence of $S = \sum_{n=2}^{\infty} \frac{(-2)^n}{\sqrt{n}}(x-3)^n$.

Solution. We first think of x as being a fixed number and apply the ratio test

$$\text{to } \sum_{n=2}^{\infty} \left| \frac{(-2)^n}{\sqrt{n}}(x-3)^n \right|:$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\left| \frac{(-2)^{n+1}}{\sqrt{n+1}}(x-3)^{n+1} \right|}{\left| \frac{(-2)^n}{\sqrt{n}}(x-3)^n \right|} &= \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}\sqrt{n}(x-3)^{n+1}}{(-2)^n\sqrt{n+1}(x-3)^n} \right| \\ &= \lim_{n \rightarrow \infty} 2 \frac{\sqrt{n+1}}{\sqrt{n}} |x-3| = 2|x-3|. \end{aligned}$$

By the Ratio Test, if $2|x-3| < 1$, then $\sum_{n=2}^{\infty} \left| \frac{(-2)^n}{\sqrt{n}}(x-3)^n \right|$ converges, and thus

S converges as well. We have

$$\begin{aligned} 2|x-3| < 1 &\iff |x-3| < 1/2 \iff -1/2 < x-3 < 1/2 \\ &\iff 5/2 < x < 7/2. \end{aligned}$$

When $x < 5/2$ or $x > 7/2$, S diverges by the n -th term test for divergence. It remains to check convergence of S at $x = 5/2$ and $x = 7/2$.

When $x = 5/2$,

$$\begin{aligned} S &= \sum_{n=2}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{5}{2} - 3 \right)^n = \sum_{n=2}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{-1}{2} \right)^n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} ((-2)(-1/2))^n \\ &= \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} (1)^n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}, \end{aligned}$$

which diverges by the p -series test, $p = 1/2$.

When $x = 7/2$,

$$\begin{aligned} S &= \sum_{n=2}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{7}{2} - 3 \right)^n = \sum_{n=2}^{\infty} \frac{(-2)^n}{\sqrt{n}} \left(\frac{1}{2} \right)^n = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} ((-2)(1/2))^n \\ &= \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \end{aligned}$$

which converges by the Alternate Series Test.

Therefore IOC = $(5/2, 7/2]$ and ROC = $1/2$.

4. (15 points) Find the third order Taylor polynomial of $f(x) = \sqrt{x}$ at $a = 1$.

Solution.

$$\begin{aligned} f(x) &= x^{1/2} & f(1) &= (1)^{1/2} = 1 \\ f'(x) &= (1/2)x^{-1/2} & f'(1) &= (1/2)(1)^{-1/2} = 1/2 \\ f''(x) &= (-1/4)x^{-3/2} & f''(1) &= (-1/4)(1)^{-3/2} = -1/4 \\ f'''(x) &= (3/8)x^{-5/2} & f'''(1) &= (3/8)(1)^{-5/2} = 3/8 \end{aligned}$$

Therefore

$$\begin{aligned} P_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 1 + 1/2(x-1) + \frac{-1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \end{aligned}$$