Math 4000/6000 Exam 1 Take Home February 28, 2007

Name:______________________________________ (100 points total)

1. (43 points) Determine and state whether each of the following statements is True or False. If True, prove briefly. If False, give a counterexample.

(a) Let $a, b, m, n \in \mathbb{N}$, and let $m \mid n$. If $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.

(b) Let $a, b, m, n \in \mathbb{N}$, and let $m \mid n$. If $a \equiv b \pmod{m}$, then $a \equiv b \pmod{n}$.

(c) Let $a, b, d \in \mathbb{Z}$. If d divides ab but d does not divide a, then d divides b.

(d) Let R be a commutative ring which is not a domain. Then R has no units other than 1 and -1.

(e) Let R be a ring, and let $a \in R$. If $ab = b$ for all $b \in R$, then $a = 1$.

(f) If R is a ring, then every element of R is either a unit or a zero-divisor.

(g) If R is a ring, then there is no element of R which is both a unit and a zero-divisor.

2. (15 points) Do there exist positive integers m and n such that $(76)^m - (34)^n = 10398457338$? Prove your answer. Hint: work mod 5.

3. (20 points) Let a, b, and c be positive integers. Without using the Fundamental Theorem of Arithmetic, prove that if $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

4. (22 points) (a) Find $\gcd(46, 297)$.
(b) Do there exist integers m and n such that $46m + 297n = -4$? If so, then find them. If not, then explain why not.
(c) Is $\overline{46}$ a unit in $\mathbb{Z}_{297}$? Is $\overline{297}$ a unit in $\mathbb{Z}_{46}$? Justify both answers.