Math 4000/6000 Exam 1 February 26, 2007

Name: **Solutions** (100 points total)

1. (43 points) Determine and state whether each of the following statements is True or False. If True, prove briefly. If False, give a counterexample.

(a) Let $a, b, m, n \in \mathbb{N}$, and let $m \mid n$. If $a \equiv b \pmod{n}$, then $a \equiv b \pmod{m}$.

> **Solution**. True. Since $a \equiv b \pmod{n}$, $n \mid (b-a)$. Thus $m \mid n$ and $n \mid (b-a)$ implies that $m \mid (b-a)$. Therefore $a \equiv b \pmod{m}$.

(b) Let $a, b, m, n \in \mathbb{N}$, and let $m \mid n$. If $a \equiv b \pmod{m}$, then $a \equiv b \pmod{n}$.

> **Solution**. False. Counterexample: $3 \equiv 8 \pmod{5}$, but $3 \not\equiv 8 \pmod{10}$.

(c) Let $a, b, d \in \mathbb{Z}$. If d divides ab but d does not divide a, then d divides b.

> **Solution**. False. Counterexample: 12 divides $3 \cdot 4$, but 12 does not divide 3 and 12 does not divide 4.

(d) Let R be a commutative ring which is not a domain. Then R has no units other than 1 and -1.

> **Solution**. False. Counterexample: $\mathbb{Z}_{14}$ is a commutative ring which is not a domain. However, $\overline{3}$ is a unit in $\mathbb{Z}_{14}$, since $\overline{3} \cdot \overline{5} = \overline{1}$.

(e) Let R be a ring, and let $a \in R$. If $ab = b$ for all $b \in R$, then $a = 1$.

> **Solution**. True. Since $ab = b$ for all $b \in R$, $ab = b$ when $b = 1$, i.e., $a \cdot 1 = 1$. However, since 1 is the multiplicative identity, $a \cdot 1 = a$. Therefore $a = 1$ (since we've shown that both a and 1 equal $a \cdot 1$).

(f) If R is a ring, then every element of R is either a unit or a zero-divisor.

> **Solution**. False. Counterexample: $2 \in \mathbb{Z}$ is neither a unit nor a zero-divisor in $\mathbb{Z}$.

(g) If R is a ring, then there is no element of R which is both a unit and a zero-divisor.

> **Solution**. True. Let $a \in R$ be a unit, so that $a^{-1} \in R$. Suppose that $ab = 0$. Then $a^{-1}(ab) = a^{-1}(0)$, which implies that $b = 0$. Similarly, $ba = 0$ implies that $b = 0$. Therefore a is not a zero-divisor.

2. (15 points) Do there exist positive integers m and n such that $(76)^m - (34)^n = 10398457338$? Prove your answer. Hint: work mod 5.

Solution. No. $76 \equiv 1 \pmod 5$ and $34 \equiv -1 \pmod 5$. Therefore $(76)^m - (34)^n \equiv (1)^m - (-1)^n \equiv 1 \pm 1 \pmod 5 \equiv 2$ or $0 \pmod 5$. However, $10398457338 \equiv 3 \pmod 5$, since it ends in 8. Therefore $(76)^m - (34)^n$ can't equal 10398457338, since the two numbers don't agree mod 5.

3. (20 points) Let a, b, and c be positive integers. Without using the Fundamental Theorem of Arithmetic, prove that if $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Solution. Since $\gcd(a, b) = 1$,

$$ma + nb = 1$$

for some $m, n \in \mathbb{Z}$. Multiplying both sides of this equation by c, we obtain

$$mac + nbc = c.$$

Clearly $a \mid mac$. Also, by assumption, $a \mid bc$, and thus $a \mid nbc$. Therefore a divides the $mac + nbc$, which equals c.

4. (22 points) (a) Find $\gcd(297, 46)$.

Solution.

$$\begin{aligned} 297 &= 46 \cdot 6 + 21 \\ 46 &= 21 \cdot 2 + 4 \\ 21 &= 4 \cdot 5 + 1 \end{aligned}$$

Therefore $\gcd(297, 46) = 1$.

(b) Do there exist integers m and n such that $297m + 46n = -4$? If so, then find them. If not, then explain why not.

Solution. Yes.

$$\begin{aligned} 1 &= 21 - 4 \cdot 5 \\ &= 21 - (46 - 21 \cdot 2)5 = 46 \cdot -5 + 21 \cdot 11 \\ &= 46 \cdot -5 + (297 - 46 \cdot 6)11 = 297 \cdot 11 - 46 \cdot 71 \end{aligned}$$

So $-4 = -4(1) = -4(297 \cdot 11 - 46 \cdot 71) = 297 \cdot -44 + 46 \cdot 284$.

(c) Is $\overline{46}$ a unit in $\mathbb{Z}_{297}$? Is $\overline{297}$ a unit in $\mathbb{Z}_{46}$? Justify both answers.

Solution. Yes and yes. Since $1 = 297 \cdot 11 - 46 \cdot 71$, it follows that $\overline{1} = \overline{-46} \cdot \overline{71}$ in $\mathbb{Z}_{297}$ and $\overline{1} = \overline{297} \cdot \overline{11}$ in $\mathbb{Z}_{46}$.