Math 4000/6000 Exam 3 April 25, 2007

Name:____ (100 points total)

1. (24 points) (a) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^3 + x^2 + 1 \in \mathbb{Q}[x]$. Explain **briefly** why any nonzero element of $\mathbb{Q}[\alpha]$ has a multiplicative inverse in $\mathbb{Q}[\alpha]$.

(b) In particular, $4 + \alpha$, an element of $\mathbb{Q}[\alpha]$, must have a multiplicative inverse in $\mathbb{Q}[\alpha]$. Find this multiplicative inverse. Your answer must be a polynomial expression in α.

2. (52 points) Determine and state whether each of the following statements is True or False. If True, then prove. If False, then **give a counterexample**.

(a) If p and q are any prime numbers and $p \neq q$, then $x^5 - p^5 q \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$.

(b) Let $f(x) \in \mathbb{Z}[x]$, let p be a prime number, and let $\overline{f}(x) \in \mathbb{Z}_p$ be the polynomial obtained by reducing each of the coefficients of $f(x)$ mod p. If $\overline{f}(x)$ is not irreducible in $\mathbb{Z}_p[x]$, then $f(x)$ is not irreducible in $\mathbb{Q}[x]$.

(c) If $\alpha, \beta \in \mathbb{C}$, then $\mathbb{Q}[\alpha, \beta] \subset \mathbb{Q}[\alpha + \beta, \alpha - \beta]$.

3. (24 points) Let R and S be commutative rings, and let $\phi : R \to S$ be a ring homomorphism. Prove that $\ker(\phi)$ is an ideal.