Math 4000/6000 Exam 3 April 26, 2007

Name: **Solutions** (100 points total)

1. (24 points) (a) Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^3 + x^2 + 1 \in \mathbb{Q}[x]$. Explain **briefly** why any nonzero element of $\mathbb{Q}[\alpha]$ has a multiplicative inverse in $\mathbb{Q}[\alpha]$.

> **Solution**. Since α is a root of $x^3 + x^2 + 1 \in \mathbb{Q}[x]$, α is algebraic over $\mathbb{Q}$. Thus, by a proposition we proved in class, $\mathbb{Q}[\alpha]$ is a field.

(b) In particular, $4 + \alpha$, an element of $\mathbb{Q}[\alpha]$, must have a multiplicative inverse in $\mathbb{Q}[\alpha]$. Find this multiplicative inverse. Your answer must be a polynomial expression in α.

Solution. Dividing $x^3 + x^2 + 1$ by $x + 4$, we obtain

$$x^3 + x^2 + 1 = (x + 4)(x^2 - 3x + 12) - 47.$$

Rearranging:

$$1 = (x + 4)\left(\frac{1}{47}x^2 - \frac{3}{47}x + \frac{12}{47}\right) - (1/47)(x^3 + x^2 + 1).$$

Substituting α for x:

$$\begin{aligned} 1 &= (\alpha + 4)\left(\frac{1}{47}\alpha^2 - \frac{3}{47}\alpha + \frac{12}{47}\right) - (1/47)(\alpha^3 + \alpha^2 + 1) \\ &= (\alpha + 4)\left(\frac{1}{47}\alpha^2 - \frac{3}{47}\alpha + \frac{12}{47}\right), \end{aligned}$$

since α is a root of $x^3 + x^2 + 1$. Therefore

$$(\alpha + 4)^{-1} = \left(\frac{1}{47}\alpha^2 - \frac{3}{47}\alpha + \frac{12}{47}\right).$$

2. (52 points) Determine and state whether each of the following statements is True or False. If True, then prove. If False, then **give a counterexample**.

(a) If p and q are any prime numbers and $p \neq q$, then $x^5 - p^5 q \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$.

Solution. True. Apply Eisenstein's criterion, with prime number q.

(b) Let $f(x) \in \mathbb{Z}[x]$, let p be a prime number, and let $\overline{f}(x) \in \mathbb{Z}_p$ be the polynomial obtained by reducing each of the coefficients of $f(x)$ mod p. If $\overline{f}(x)$ is not irreducible in $\mathbb{Z}_p[x]$, then $f(x)$ is not irreducible in $\mathbb{Q}[x]$.

Solution. False. Counterexample: $f(x) = x^2 + 3$. $\overline{f}(x) = x^2 + \overline{1} \in \mathbb{Z}_2[x]$, which is not irreducible: $x^2 + \overline{1} = (x + \overline{1})(x + \overline{1})$. However, $f(x)$ is irreducible in $\mathbb{Q}[x]$ by Eisenstein's criterion, with prime number 3.

(c) If $\alpha, \beta \in \mathbb{C}$, then $\mathbb{Q}[\alpha, \beta] \subset \mathbb{Q}[\alpha + \beta, \alpha - \beta]$.

Solution. True.

$$\begin{aligned} \alpha &= (1/2)(\alpha + \beta) + (1/2)(\alpha - \beta) \in \mathbb{Q}[\alpha + \beta, \alpha - \beta] \\ \beta &= (1/2)(\alpha + \beta) - (1/2)(\alpha - \beta) \in \mathbb{Q}[\alpha + \beta, \alpha - \beta] \end{aligned}$$

Therefore $\mathbb{Q}[\alpha, \beta] \subset \mathbb{Q}[\alpha + \beta, \alpha - \beta]$.

3. (24 points) Let R and S be commutative rings, and let $\phi : R \to S$ be a ring homomorphism. Prove that $\ker(\phi)$ is an ideal.

Solution. $\ker(\phi)$ is nonempty, since $0 \in \ker(\phi)$. If $a, b \in \ker(\phi)$ and $r \in R$, then

- $a + b \in \ker(\phi)$, since $\phi(a + b) = \phi(a) + \phi(b) = 0 + 0 = 0$.
- $ra \in \ker(\phi)$, since $\phi(ra) = \phi(r)\phi(a) = \phi(r)0 = 0$.

Therefore $\ker(\phi)$ is an ideal.