Homework 3: Permutations and Combinations of Sets

1. Brualdi, Chapter 3, #7.

2. Brualdi, Chapter 3, #11.

3. Brualdi, Chapter 3, #13.

4. Brualdi, Chapter 3, #28.

5. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{3, 6, 9, \ldots, 99\}$. A function f from A to B is said to be *increasing* if $a < b$ implies $f(a) < f(b)$.
(a) How many increasing functions are there from A to B?
(b) How many increasing functions f from A to B satisfy $f(5) < 24$?
(c) How many increasing functions f from A to B satisfy $f(4) = 21$ and $f(7) = 63$?
(d) How many functions f from A to B satisfy $f(1) < f(2) < f(3)$ and $f(6) > f(7) > f(8)$?
(e) How many of the functions of part (d) are one-to-one?

Challenge problems (students enrolled in Math 6670 or CSCI 6670 should turn in both of the following problems).

6. How many 15 element subsets of $\{1, 2, 3, \ldots, 40\}$ are there which do not contain two consecutive integers?

7. Let $S = \{1, 2, \ldots, n+1\}$, where $n \geq 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \text{ and } y < z\}.$$

(a) Show by counting $|T|$ in two different ways that

$$\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

(b) Use the equation in (a) to express $\sum_{k=0}^{n} k^2$ as a polynomial in n.

For those interested in exploring this idea further: can you derive a closed formula for $\sum_{k=0}^{n} k^3$ using a similar method? How about $\sum_{k=0}^{n} k^p$, for arbitrary positive integer p?