

Homework 5: Distribution Problems and the Binomial Theorem

1. Brualdi, Chapter 3, #36.
2. Brualdi, Chapter 3, #42.
3. Brualdi, Chapter 3, #43.
4. Brualdi, Chapter 5, #4.
5. Brualdi, Chapter 5, #6.
6. Brualdi, Chapter 5, #7.
7. Brualdi, Chapter 5, #16.
8. Brualdi, Chapter 5, #24.
9. How many integer solutions of $x_1 + x_2 + x_3 + x_4 = 45$ satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq 3$, and $x_4 \geq 8$?

10. The number of ways to distribute p distinguishable objects among j indistinguishable boxes with no box empty is denoted by $\left\{ \begin{smallmatrix} p \\ j \end{smallmatrix} \right\}$, and is called a *Stirling number of the second kind*.

(a) Prove that for any $p \geq 3$, $\left\{ \begin{smallmatrix} p \\ p \end{smallmatrix} \right\} = 1$, $\left\{ \begin{smallmatrix} p \\ p-1 \end{smallmatrix} \right\} = \binom{p}{2}$, and $\left\{ \begin{smallmatrix} p \\ p-2 \end{smallmatrix} \right\} = \binom{p}{3} + 3\binom{p}{4}$. You may *not* use any known formulas for $\left\{ \begin{smallmatrix} p \\ j \end{smallmatrix} \right\}$, such as the one given in the remark below.

(b) Explain why the number of ways to distribute p distinguishable objects among j distinguishable boxes with no box empty is $j! \left\{ \begin{smallmatrix} p \\ j \end{smallmatrix} \right\}$.

Remark: In class we will soon prove that, for all nonnegative integers p and j , $\left\{ \begin{smallmatrix} p \\ j \end{smallmatrix} \right\} = \frac{1}{j!} \sum_{i=0}^j (-1)^i \binom{j}{i} (j-i)^p$ (by using the Principle of Inclusion and Exclusion).

Challenge problem (students enrolled in Math 6670 should turn in Problem 11).

11. In this exercise we will prove that for any positive integers n and p ,

$$\sum_{k=0}^n k^p = \sum_{j=1}^p j! \left\{ \begin{smallmatrix} p \\ j \end{smallmatrix} \right\} \binom{n+1}{j+1}. \quad (1)$$

(a) Let $S = \{1, 2, \dots, n+1\}$, and let

$$T = \{(x_1, x_2, \dots, x_{p+1}) \in S^{p+1} \mid x_1 < x_{p+1}, x_2 < x_{p+1}, \dots, x_p < x_{p+1}\}.$$

Prove (1) by counting $|T|$ in two different ways. (Hint: use 10(b).)

(b) Use (1) to express $\sum_{k=0}^n k^3$ as a polynomial in n of degree 4. (Hint: use 10(a) or the formula given in the remark below problem 10, whichever you prefer.)

(c) Use (1) to explain why $\sum_{k=0}^n k^p$ can be expressed as a polynomial in n of degree $p+1$. What is the coefficient of the n^{p+1} term?