Homework 7: The Inclusion-Exclusion Principle

1. Find the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 50$ such that $1 \leq x_1 \leq 15$, $0 \leq x_2 \leq 20$, $7 \leq x_3 \leq 25$, and $0 \leq x_4$.

2. Determine the number of ways to distribute 35 distinguishable objects among 20 distinguishable boxes if
(a) at least 1 box must be empty.
(b) at most 3 boxes can be empty.

3. Each of n children brings one parent to parent-teacher conferences. Determine the number of ways can these $2n$ people be divided into n groups of 2 if
(a) there are no further restrictions on how these groups can be formed.
(b) each group must consist of one parent and one child.
(c) no child can be in the same group with his or her parent.
(d) each group must consist of one parent and one child, and no child can be in the same group with his or her parent.

4. Each of n children brings one mother, one father, and a grandparent to parent-teacher conferences. How many ways can these $4n$ people be divided into n groups if each group must consist of one child, one mother, one father, and one grandparent, and no group can be comprised of all 4 people of the same family?

5. How many ways can the 5 vertices of a pentagon be colored, if n different colors are available, and no two vertices joined by an edge can be the same color?

6. How many functions are there from a set of 13 objects to a set of 8 objects? How many of these functions are one-to-one? How many are onto?

Challenge problems (students enrolled in Math 6670 should turn in 7, 10, and 11).

7. Brualdi, Chapter 6, #31.

8. Use the formula for $\varphi(n)$ you proved in Problem 7 to prove that if m and n are relatively prime, then $\varphi(mn) = \varphi(m)\varphi(n)$.

9. Find all numbers n such that $\varphi(n) = 12$, 13, or 14.

10. Let A be a subset of B. Let $|A| = m$, $|B| = n$, and let r be some integer $m \leq r \leq n$. Show that the number of r-element subsets of B which contain A as a subset is $\binom{n-m}{r-m} = \binom{n-m}{n-r}$.

11. Use the Inclusion-Exclusion Principle to show that for any nonnegative integers m, r, n such that $m \leq r \leq n$,

$$\binom{n-m}{n-r} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \binom{n-i}{r}.$$