Homework 9: Recurrence Relations

1. Brualdi, Chapter 7, #20

2. Brualdi, Chapter 7, #21

In problems 3, 4, 6, and 7, you are asked to find recurrence relations. All of your answers should be of order less than or equal to 3. For each recurrence relation, state for which values of n the recurrence relation is valid, and give the initial conditions.

3. Find a recurrence relation for h_n, the number of ways to tile a $2 \times n$ rectangle, if you are allowed to use
(a) 1×2 tiles.
(b) 2×2 tiles and 1×2 tiles.
(c) 1×1 tiles and L-tiles, where an L-tile is a 2×2 square with the upper right 1×1 square deleted.

4. Find a recurrence relation for h_n, the number of n-digit ternary strings
(a) in which no 0 precedes a 1 (i.e., no 0 lies anywhere to the left of any 1).
(b) with an even number of 0's. Recall that 0 is an even number.
(c) with an odd number of 0's.
(d) which contain neither the substring 01 nor the substring 10 (i.e., no 0 is immediately adjacent to any 1).
(e) which do not contain the substring 000 (i.e., which don't contain three consecutive 0's).
(f) which do not contain the substring 010.

5. Let a_n denote the number of n-digit ternary strings with an even number of 0's, and let b_n denote the number with an odd number of 0's. Explain why $a_n + b_n = 3^n$. Use 4(b) and 4(c) to find another relationship between a_n and b_n. (Can you see why this relationship is true without using 4(b) and 4(c)?) Use these two relationships to find a_n and b_n, and thus solve the recurrence relations of 4(b) and 4(c).

6. Find a recurrence relation for h_n, the number of ways to write n as an ordered sum of one or more positive integers, where each summand is at least 2. (For example, $h_5 = 3$, since we may write 5 as 5, $3 + 2$, or $2 + 3$.)

7. Let n lines be drawn in the plane such that no two lines are parallel and no three lines meet in a single point (thus the lines divide the plane into a maximal number of regions, as discussed in class). Find a recurrence relation for h_n, the number of *infinite* regions that result.

8. Solve the recurrence relations of 3(b), 4(a), 4(d), 6, and 7.

Challenge problem (students enrolled in Math 6670 should turn in Problem 9).

9. Let D_n denote the number of derangements of n objects. Give two proofs that D_n satisfies the following recurrence relation:

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}.$$

For one proof, use the formula derived in class for D_n. For the other proof, give a combinatorial argument.