Math 2250 Exam 1 February 7, 2008

Name: Solutions (100 points total)

1. (25 points) Let $f(x) = \dfrac{5}{x^2}$.

(a) Using the **definition of derivative**, find the derivative of $f(x)$. Your answer should be a function of x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{5}{(x+h)^2} - \frac{5}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{5x^2}{x^2(x+h)^2} - \frac{5(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 - 5(x+h)^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{5x^2 - 5(x^2 + 2hx + h^2)}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{-10hx - 10h^2}{hx^2(x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-10x - 10h)}{\cancel{h}x^2(x+h)^2}$$

$$= \frac{-10x - 10(0)}{x^2(x+0)^2} = \frac{-10x}{x^4}$$

$$= \boxed{\frac{-10}{x^3}}$$

(b) Give an equation of the tangent line to the graph of $f(x)$ at $x = 2$.

$$f(x) = \frac{5}{x^2} \qquad f(2) = \frac{5}{2^2} = \frac{5}{4}$$

$$f'(x) = \frac{-10}{x^3} \qquad f'(2) = \frac{-10}{2^3} = \frac{-10}{8} = \frac{-5}{4}$$

slope of tangent line: $\frac{-5}{4}$

point on tangent line: $(2, \frac{5}{4})$

$\therefore$ eqn. of tangent line: $\boxed{y = \frac{5}{4} + \frac{-5}{4}(x-2)}$

2. (23 points) Find the following limits.

(a) $\lim_{x\to 3} \frac{x^2-5x+6}{x^2+x-12}$ $= \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x+4)} = \lim_{x\to 3} \frac{x-2}{x+4}$

$= \frac{3-2}{3+4} = \boxed{\frac{1}{7}}$

(b) $\lim_{x\to 2} \frac{x^2-3x+12}{x^3-3x^2+5x+1}$ $= \frac{(2)^2-3(2)+12}{(2)^3-3(2)^2+5(2)+1}$

$= \boxed{\frac{10}{7}}$

(c) $\lim_{x\to\infty} \frac{\cos 5x}{2x}$ $= \boxed{0}$

(d) $\lim_{x\to 0} \frac{x^3-2x+\sin x}{4x}$ $= \lim_{x\to 0} \frac{x^3}{4x} - \lim_{x\to 0} \frac{2x}{4x} + \lim_{x\to 0} \frac{\sin x}{4x}$

$= \lim_{x\to 0} \frac{x^2}{4} - \lim_{x\to 0} \frac{2}{4} + \frac{1}{4}\lim_{x\to 0} \frac{\sin x}{x}$

$= 0 - \frac{2}{4} + \frac{1}{4}\cdot 1 = \boxed{-\frac{1}{4}}$

3. (20 points) Consider

$$f(x) = \begin{cases} 9 - x^2, & \text{if } x < 1 \\ 4, & \text{if } x = 1 \\ x^4 + 4x + 3, & \text{if } x > 1 \end{cases}$$

(a) Is $f(x)$ defined at $x = 1$? If so, what is $f(1)$?

Yes. $f(1) = 4$.

(b) Determine whether $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$ exist, and if so compute them.

$\lim_{x \to 1^-} f(x) = 9 - (1)^2 = 8$

$\lim_{x \to 1^+} f(x) = (1)^2 + 4(1) + 3 = 8$

Both limits exist

(c) Does $\lim_{x \to 1} f(x)$ exist? Justify your answer. If the limit does exist, then what is the limit?

- $\lim_{x \to 1^-} f(x)$ exists
- $\lim_{x \to 1^+} f(x)$ exists
- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

Since $f(x)$ satisfies these three conditions, $\lim_{x \to 1} f(x)$ exists.

Its value is 8.

(d) Is $f(x)$ continuous at $x = 1$? Justify your answer.

- $f(1)$ exists ✓
- $\lim_{x \to 1} f(x)$ exists ✓
- $\lim_{x \to 1} f(x) = f(1)$ ✗

8 4

Since $f(x)$ does not satisfy the third condition, $f(x)$ is not continuous at $x=1$.

4. (25 points) Consider $f(x) = \frac{-3x^2}{x^2-4}$. $= \frac{-3x^2}{(x+2)(x-2)}$

(a) Find the equations of all horizontal and vertical asymptotes of $f(x)$.

Horizontal Asymptote

$$\lim_{x\to\infty} \frac{-3x^2}{x^2-4} = -3$$

$$\lim_{x\to-\infty} \frac{-3x^2}{x^2-4} = -3$$

$\therefore$ Horiz. Asymptote: $\boxed{y=-3}$

Vertical Asymptotes

$\boxed{x=-2, \quad x=2}$

(b) Compute the left and right hand limits at each vertical asymptote.

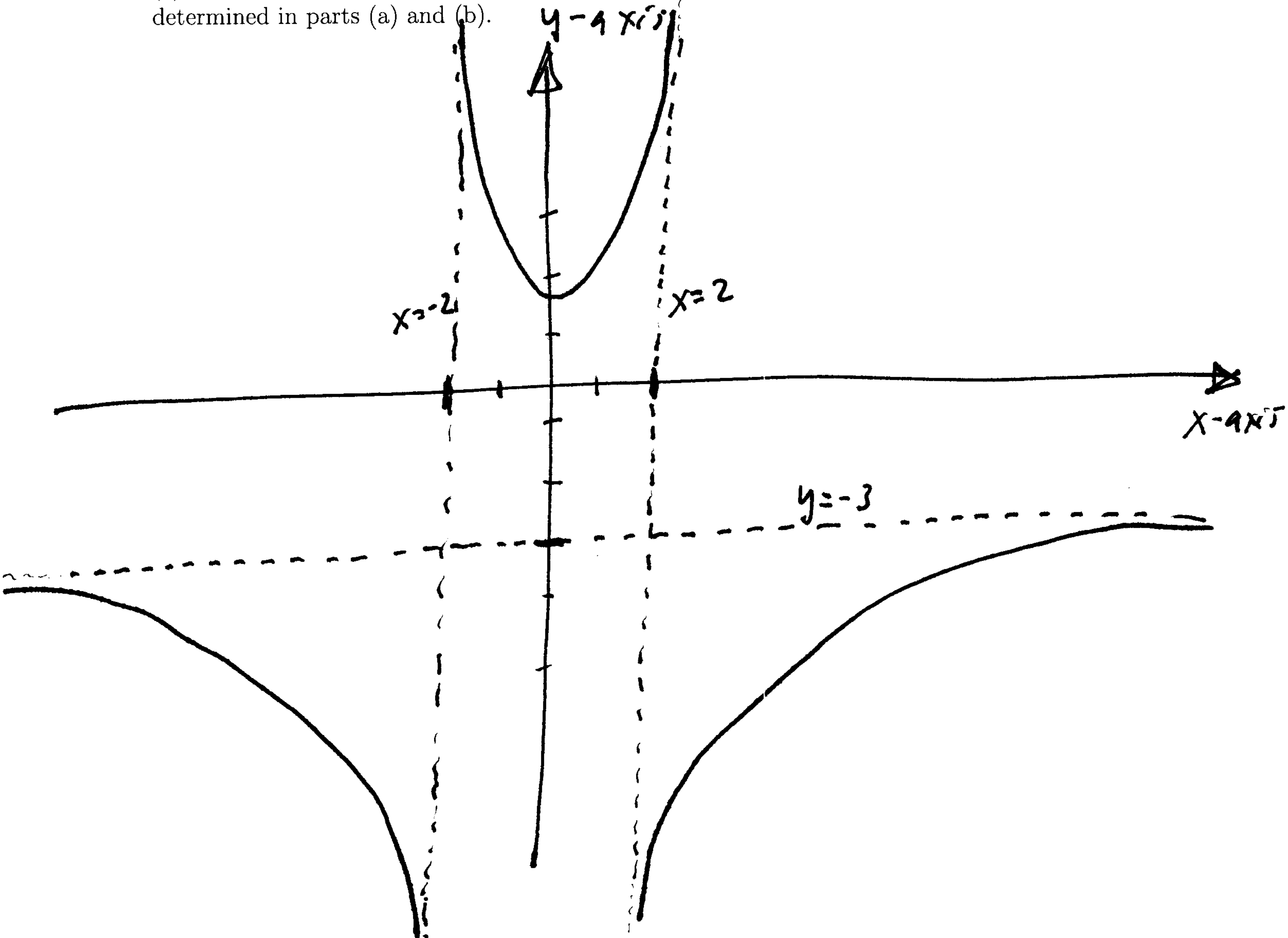
$$\lim_{x\to 2^+} \frac{-3x^2}{(x+2)(x-2)} = -\infty \qquad \frac{(-)}{(+)(+)}$$

$$\lim_{x\to 2^-} \frac{-3x^2}{(x+2)(x-2)} = +\infty \qquad \frac{-}{(+)(-)}$$

$$\lim_{x\to -2^+} \frac{-3x^2}{(x+2)(x-2)} = +\infty \qquad \frac{-}{(+)(-)}$$

$$\lim_{x\to -2^-} \frac{-3x^2}{(x+2)(x-2)} = -\infty \qquad \frac{-}{(-)(-)}$$

(c) Sketch the graph of $f(x)$. Your graph should incorporate all of the information you determined in parts (a) and (b).



5. (7 points) Let $g(x)$ be a function. In one English sentence, explain the geometrical meaning of the following mathematical statement: $g'(7) = 5$. The only mathematical symbols your sentence is allowed to contain are '$g(x)$', 'x', '7', and '5'. Your sentence should be grammatically correct.

The slope of the tangent line to the graph of g(x) at x=7 is 5.