Do not write in the boxes immediately below.

Question:	1	2	3	4	5	6	Total
Points:	36	10	10	10	16	18	100
Score:							

MATH 2250 Exam 4

April 24, 2008

Name: **Solutions**

1. (36 points) Evaluate the following:

   (a) $\displaystyle\int \left(\frac{x^2+1}{4} - \frac{4}{x^2+1}\right) dx$

Solution:

$$\begin{aligned}
\int \left(\frac{x^2+1}{4} - \frac{4}{x^2+1}\right) dx &= \int \left(\frac{x^2}{4} + \frac{1}{4} - \frac{4}{x^2+1}\right) dx \\
&= \int \frac{x^2}{4}\, dx + \int \frac{1}{4}\, dx - \int \left(\frac{4}{x^2+1}\right) dx \\
&= \frac{1}{4}\int x^2\, dx + \frac{1}{4}\int 1\, dx - 4\int \left(\frac{1}{x^2+1}\right) dx \\
&= \frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - 4\tan^{-1} x + C \\
&= \frac{x^3}{12} + \frac{x}{4} - 4\tan^{-1} x + C
\end{aligned}$$

   (b) $\displaystyle\int_0^{\pi/16} \sec^2(4x)\, dx$

Solution:

$$\begin{aligned}
\int_0^{\pi/16} \sec^2(4x)\, dx &= \frac{1}{4}\tan(4x)\Big|_0^{\pi/16} \\
&= \frac{1}{4}\tan(4(\pi/16)) - \frac{1}{4}\tan(4(0)) \\
&= \frac{1}{4}\tan(\pi/4) - \frac{1}{4}\tan(0) = \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 0 = \frac{1}{4}
\end{aligned}$$

(c) $\int_1^2 2x\left(\frac{x^2+3}{x^2}\right)\,dx$

Solution:

$$\begin{aligned}
\int_1^2 2x\left(\frac{x^2+3}{x^2}\right)\,dx &= \int_1^2 2\left(\frac{x^2+3}{x}\right)\,dx \\
&= \int_1^2 \left(\frac{2x^2+6}{x}\right)\,dx \\
&= \int_1^2 \left(\frac{2x^2}{x}+\frac{6}{x}\right)\,dx \\
&= \int_1^2 \left(2x+\frac{6}{x}\right)\,dx \\
&= \left(x^2+6\ln|x|\right)\Big|_1^2 \\
&= \left(2^2+6\ln|2|\right)-\left(1^2+6\ln|1|\right) \\
&= 4+6\ln 2-1-6\cdot 0 = 3+6\ln 2
\end{aligned}$$

2. (10 points) For each of the following, answer True or False:

(a) $\lim_{x\to 3}\left(\frac{x-3}{x^2-3}\right)=\lim_{x\to 3}\left(\frac{1}{2x}\right)=\frac{1}{6}$.

Solution: False. $\lim_{x\to 3}\left(\frac{x-3}{x^2-3}\right)=\left(\frac{(3)-3}{(3)^2-3}\right)=\frac{0}{6}=0.$

(b) For any function $f(x)$, there is at most one function $F(x)$ such that $F'(x)=f(x)$.

Solution: False. For example, let $f(x)=2x$. Then $F(x)=x^2$, $F(x)=x^2+3$, $F(x)=x^2+11$ are three distinct antiderivatives. In fact, there are an infinite number of antiderivatives, namely all functions of the form x^2+C, where C is a constant.

3. (10 points) Let $f(x) = \int_2^{\ln x} (t+1)\cos(t^2)\, dt$. Find $f'(x)$.

Solution: Let $y = \ln x$, so $f(y) = \int_2^y (t+1)\cos(t^2)\,dt$.
By the chain rule,

$$\begin{aligned}
\frac{df(y)}{dx} &= \frac{df(y)}{dy} \cdot \frac{dy}{dx} \\
&= \frac{d}{dy}\left(\int_2^y (t+1)\cos(t^2)\,dt\right) \cdot \frac{d}{dx}(\ln x) \\
&= (y+1)\cos(y^2) \cdot \frac{1}{x} \\
&= ((\ln x)+1)\cos((\ln x)^2) \cdot \frac{1}{x} \\
&= \frac{(\ln x + 1)\cos((\ln x)^2)}{x}
\end{aligned}$$

4. (10 points) Approximate $\int_{-1}^{1} x^3\,dx$ by using a Riemann sum with $n = 4$ and left endpoints.

Solution: Begin by dividing $[-1, 1]$ into 4 intervals, each of length $dx = \frac{1-(-1)}{4} = \frac{1}{2}$. Let x_1, x_2, x_3, x_4 be the left endpoints of these 4 intervals: $x_1 = -1$, $x_2 = -.5$, $x_3 = 0$, $x_4 = .5$. Let $f(x) = x^3$. Then

$$\begin{aligned}
\int_{-1}^{1} f(x)\,dx &\approx f(x_1)dx + f(x_2)dx + f(x_3)dx + f(x_4)dx \\
&= (-1)^3(.5) + (-.5)^3(.5) + (0)^3(.5) + (.5)^3(.5) \\
&= (-1)^3(.5) \\
&= -.5
\end{aligned}$$

5. (16 points) Evaluate $\lim_{x\to 0} (\cos x - \sin x)^{2/\sin x}$.

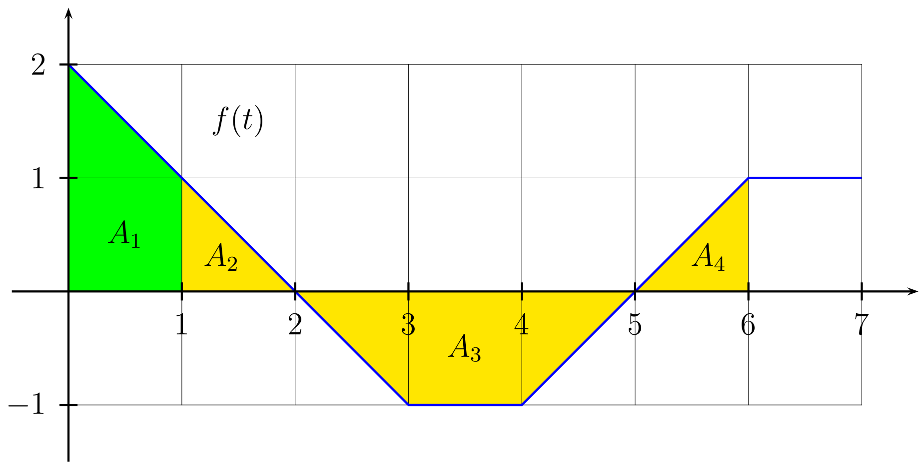
Solution: Substituting 0 for x, we observe that this limit is of the form 1^∞. Thus, our approach to solving this problem will be to first find $\lim_{x\to 0} \ln\left((\cos x - \sin x)^{2/\sin x}\right)$, and then use this result to determine $\lim_{x\to 0} (\cos x - \sin x)^{2/\sin x}$.

$$
\begin{aligned}
\lim_{x\to 0} \ln\left((\cos x - \sin x)^{2/\sin x}\right) &= \lim_{x\to 0} (2/\sin x)\ln(\cos x - \sin x) \\
&= \lim_{x\to 0} \frac{2\ln(\cos x - \sin x)}{\sin x} \qquad \text{(of type } 0/0) \\
&= \lim_{x\to 0} \frac{(2\ln(\cos x - \sin x))'}{(\sin x)'} \qquad \text{(applying L'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{\frac{2}{\cos x - \sin x}(\cos x - \sin x)'}{\cos x} \qquad \text{(applying chain rule)} \\
&= \lim_{x\to 0} \frac{\frac{2}{\cos x - \sin x}(\sin x - \cos x)}{\cos x} \\
&= \frac{\frac{2}{\cos 0 - \sin 0}(\sin 0 - \cos 0)}{\cos 0} \\
&= \frac{\frac{2}{1-0}(0-1)}{1} \\
&= -2.
\end{aligned}
$$

Since $\ln\left((\cos x - \sin x)^{2/\sin x}\right)$ approaches -2 as x approaches 0, $e^{\ln\left((\cos x - \sin x)^{2/\sin x}\right)}$ approaches e^{-2} as x approaches 0. Hence $(\cos x - \sin x)^{2/\sin x}$, which equals $e^{\ln\left((\cos x - \sin x)^{2/\sin x}\right)}$, approaches e^{-2} as x approaches 0.

Answer: $\lim_{x\to 0} (\cos x - \sin x)^{2/\sin x} = e^{-2}$.

6. (18 points) Let $f(t)$ be the function whose graph is shown below, and let $g(x) = \int_1^x f(t)\,dt$.



Compute the following:

(a) $\int_1^6 f(t)\,dt$

Solution: $\int_1^6 f(t)\,dt = A_2 - A_3 + A_4 = 0.5 - 2 + 0.5 = -1.$

(b) $\int_1^0 f(t)\,dt$

Solution: $\int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = -A_1 = -1.5.$

(c) $g'(2)$

Solution: $g'(x) = \frac{d}{dx}\,g(x) = \frac{d}{dx}\int_1^x f(t)\,dt = f(x)$. Therefore $g'(2) = f(2) = 0$.

(d) $g'(4)$

Solution: $g'(4) = f(4) = -1.$