MATH 2250 Exam 1

February 10, 2009

Name: **Solutions**

1. (23 points) Let $f(x) = 5x + 3\sqrt{x}$. Using the **definition of derivative**, find the derivative of $f(x)$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\big(5(x+h) + 3\sqrt{x+h}\big) - \big(5x + 3\sqrt{x}\big)}{h} \\
&= \lim_{h\to 0} \frac{5x + 5h + 3\sqrt{x+h} - 5x - 3\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{5h + 3\sqrt{x+h} - 3\sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{5h}{h} + \frac{3\sqrt{x+h} - 3\sqrt{x}}{h}\right) \\
&= \lim_{h\to 0} \left(5 + \frac{3\sqrt{x+h} - 3\sqrt{x}}{h}\right) \\
&= 5 + \lim_{h\to 0} \frac{3\sqrt{x+h} - 3\sqrt{x}}{h} \\
&= 5 + 3\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= 5 + 3\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= 5 + 3\lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= 5 + 3\lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= 5 + 3\lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= 5 + 3\frac{1}{\sqrt{x+0} + \sqrt{x}} \\
&= 5 + 3\frac{1}{2\sqrt{x}} \\
&= 5 + \frac{3}{2\sqrt{x}}
\end{aligned}
$$

2. (36 points) Compute each of the following limits. If a limit does not exist, write DNE, and briefly explain why it does not exist.

(a) $\lim_{x\to\infty} \frac{\sin x}{x}$

Solution: $\lim_{x\to\infty} \frac{\sin x}{x} = 0$

(b) $\lim_{x\to 0^-} \frac{4|x|+x}{5x-2|x|}$

Solution: Recall that $|x| = -x$ when $x < 0$. Therefore

$$\begin{aligned}
\lim_{x\to 0^-} \frac{4|x|+x}{5x-2|x|} &= \lim_{x\to 0^-} \frac{4(-x)+x}{5x-2(-x)} \\
&= \lim_{x\to 0^-} \frac{-3x}{7x} \\
&= \lim_{x\to 0^-} \frac{-3}{7} \\
&= \frac{-3}{7}
\end{aligned}$$

(c) $\lim_{x\to\infty} \frac{9x^3+3x^2+2009}{x^4-3x}$

Solution: $\lim_{x\to\infty} \frac{9x^3+3x^2+2009}{x^4-3x} = \lim_{x\to\infty} \frac{9x^3}{x^4} = \lim_{x\to\infty} \frac{9}{x} = 0$

(d) $\lim_{x\to 4} \frac{x^2+2}{(x-1)(x-4)}$

Solution: One can check that

$$\lim_{x\to 4^-} \frac{x^2+2}{(x-1)(x-4)} = -\infty$$
$$\lim_{x\to 4^+} \frac{x^2+2}{(x-1)(x-4)} = \infty$$

Since the left-hand and right-hand limits are not the same,

$$\lim_{x\to 4} \frac{x^2+2}{(x-1)(x-4)} \quad \text{DNE (Does Not Exist).}$$

(e) $\lim_{x\to 0} \frac{\sin x}{3x\cos x}$ (Hint: $\frac{\sin x}{3x\cos x} = \frac{\sin x}{x} \cdot \frac{1}{3\cos x}$)

Solution:

$$\lim_{x\to 0} \frac{\sin x}{3x\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{3\cos x} = \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{3\cos x}\right)$$
$$= (1)\left(\frac{1}{3\cos 0}\right) = (1)\left(\frac{1}{3\cdot 1}\right) = \frac{1}{3}$$

(f) $\lim_{x\to 3^+} \frac{x^2-1}{(x-3)(x-5)}$

Solution: $\lim_{x\to 3^+} \frac{x^2-1}{(x-3)(x-5)} = -\infty$

3. (14 points) Consider the function

$$f(x) = \begin{cases} x^3 - 5x, & \text{if } x < -2 \\ -3x+1, & \text{if } -2 \le x < 1 \\ x^5 - 4x + 1, & \text{if } 1 \le x \end{cases}$$

(a) Circle the first true statement:

1. $f(x)$ is not defined at $x = -2$.
2. $\lim_{x\to -2} f(x)$ does not exist.
3. $\lim_{x\to -2} f(x) \neq f(-2)$.
4. $f(x)$ is continuous at $x = -2$.

Solution: $\lim_{x\to -2} f(x)$ does not exist since $\lim_{x\to -2^-} f(x) = 2$ and $\lim_{x\to -2^+} f(x) = 7$.

(b) Circle the first true statement:

1. $f(x)$ is not defined at $x = 1$.
2. $\lim_{x\to 1} f(x)$ does not exist.
3. $\lim_{x\to 1} f(x) \neq f(1)$.
4. $f(x)$ is continuous at $x = 1$.

Solution: $f(1) = -2$ and $\lim_{x\to 1} f(x) = -2$. Thus $f(x)$ is continuous at $x = 1$.

4. (14 points) Consider $f(x) = \dfrac{2x^2 - 2x - 12}{x^2 - 7x + 12}$.

   (a) Compute $\lim_{x \to 3} f(x)$. If the limit does not exist, write DNE, and briefly explain why it does not exist.

   Solution: Note that this limit is of type $\frac{0}{0}$.

$$\lim_{x\to 3} \frac{2x^2 - 2x - 12}{x^2 - 7x + 12} = \lim_{x \to 3} \frac{2(x-3)(x+2)}{(x-3)(x-4)} = \lim_{x\to 3} \frac{2(x+2)}{(x-4)} = \frac{2((3)+2)}{((3)-4)} = \frac{10}{-1} = -10$$

   (b) Find all vertical and horizontal asymptotes of $f(x)$.

   Solution: We have

$$\lim_{x\to\infty} \frac{2x^2 - 2x - 12}{x^2 - 7x + 12} = \lim_{x\to\infty} \frac{2x^2}{x^2} = \lim_{x\to\infty} 2 = 2.$$

   Similarly,

$$\lim_{x\to-\infty} \frac{2x^2 - 2x - 12}{x^2 - 7x + 12} = \lim_{x\to-\infty} \frac{2x^2}{x^2} = \lim_{x\to-\infty} 2 = 2.$$

   Thus $f(x)$ has horizontal asymptote $y = 2$. It has vertical asymptote $x = 4$.

   Note: As computed in part (a), $\lim_{x\to 3} f(x) = -10$, a finite number. Therefore $x = 3$ is not a vertical asymptote of $f(x)$.

5. (13 points) Suppose that a function $f(x)$ has derivative $f'(x) = 2/(x+4)$, and suppose also that $f(2) = 5$. Find an equation for the tangent line to the graph of $f(x)$ at $x = 2$.

   Solution: We are given that $f(2) = 5$, and we compute $f'(2) = 2/((2)+4) = 1/3$. Therefore the tangent line to the graph of $f(x)$ at $x = 2$ passes through the point $(2, 5)$ and has slope $1/3$. So its equation is $y = 5 + (1/3)(x - 2)$.