MATH 2250 Exam 2

March 19, 2009

Name: **Solutions**

1. Find the derivatives of the following functions. Do not simplify your answers.

   (a) (7 points) $f(x) = 2\sin^{-1}(x^2+3x)$

   Solution:

$$\begin{aligned} f'(x) &= 2\frac{1}{\sqrt{1-(x^2+3x)^2}}(x^2+3x)' \\ &= 2\frac{1}{\sqrt{1-(x^2+3x)^2}}(2x+3) \end{aligned}$$

   (b) (11 points) $f(x) = \cos^4\left(\frac{x^2+1}{x^2-1}\right)$

   Solution: $f(x) = \left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)^4$. Therefore

$$\begin{aligned} f'(x) &= 4\left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)^3\left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)' \\ &= 4\left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)^3\left(-\sin\left(\frac{x^2+1}{x^2-1}\right)\right)\left(\frac{x^2+1}{x^2-1}\right)' \\ &= -4\left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)^3\left(\sin\left(\frac{x^2+1}{x^2-1}\right)\right)\left(\frac{(x^2+1)'(x^2-1)-(x^2+1)(x^2-1)'}{(x^2-1)^2}\right) \\ &= -4\left(\cos\left(\frac{x^2+1}{x^2-1}\right)\right)^3\left(\sin\left(\frac{x^2+1}{x^2-1}\right)\right)\left(\frac{(2x)(x^2-1)-(x^2+1)(2x)}{(x^2-1)^2}\right) \end{aligned}$$

(c) (10 points) $f(x) = 4x \tan x \sec x$

Solution: $f(x) = (4x \tan x)(\sec x)$. Therefore

$$\begin{aligned} f'(x) &= (4x \tan x)'(\sec x) + (4x \tan x)(\sec x)' \\ &= ((4)(\tan x) + (4x)(\sec^2 x))(\sec x) + (4x \tan x)(\sec x \tan x) \end{aligned}$$

(d) (15 points) $f(x) = 7(x^3 + 5)^{\ln x}$

Solution: Let $y = 7(x^3 + 5)^{\ln x}$.

Take ln of both sides of equation:

$$\ln(y) = \ln\left(7(x^3 + 5)^{\ln x}\right)$$

$$\ln(y) = \ln 7 + \ln\left((x^3 + 5)^{\ln x}\right)$$

$$\ln(y) = \ln 7 + \ln x \ln(x^3 + 5)$$

Take $\dfrac{d}{dx}$ of both sides of equation:

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}(\ln 7 + \ln x \ln(x^3 + 5))$$

$$\frac{1}{y}\frac{dy}{dx} = 0 + (\ln x)'(\ln(x^3 + 5)) + (\ln x)(\ln(x^3 + 5))'$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x}\right)(\ln(x^3 + 5)) + (\ln x)\left(\frac{1}{x^3 + 5}(x^3 + 5)'\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x}\right)(\ln(x^3 + 5)) + (\ln x)\left(\frac{1}{x^3 + 5}(3x^2)\right)$$

Multiply both sides of equation by y:

$$\frac{dy}{dx} = y\left(\left(\frac{1}{x}\right)(\ln(x^3 + 5)) + (\ln x)\left(\frac{1}{x^3 + 5}(3x^2)\right)\right)$$

Substitute $y = 7(x^3 + 5)^{\ln x}$:

$$\frac{dy}{dx} = 7(x^3 + 5)^{\ln x}\left(\left(\frac{1}{x}\right)(\ln(x^3 + 5)) + (\ln x)\left(\frac{1}{x^3 + 5}(3x^2)\right)\right)$$

(e) (3 points) $f(x) = e^8 + 3\ln(5^2)$

Solution: Since $f(x)$ is a constant, $f'(x) = 0$.

2. (15 points) Find an equation of the tangent line to the parametric curve

$$x = t^3 - 5t$$
$$y = 7t^2 - t^4$$

at the point where $t = 2$.

Solution:

Point on tangent line:

$$((2)^3 - 5(2), 7(2)^2 - (2)^4) = (-2, 12)$$

Slope of tangent line:

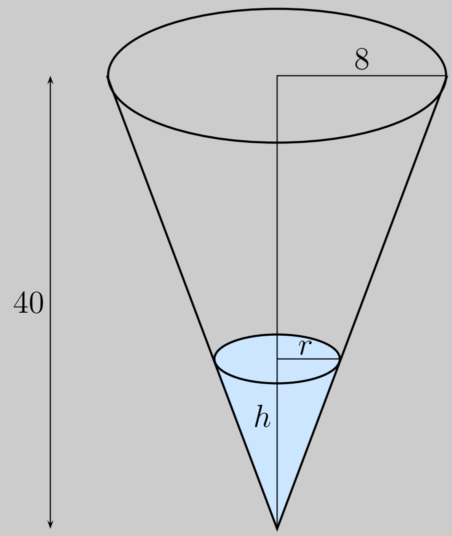
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{14t - 4t^3}{3t^2 - 5}$. Therefore at $t = 2$, $\frac{dy}{dx} = \frac{14(2) - 4(2)^3}{3(2)^2 - 5} = \frac{-4}{7}$.

Equation of tangent line:

$$y = 12 + (-4/7)(x - (-2))$$

3. (15 points) A conical water tank with vertex down has a radius of 8 feet at the top and is 40 feet high. Suppose that water is being drained from the tank. At a certain time, the water is 10 feet deep, and the radius of the water's surface is decreasing by 3 feet per second. How fast is the volume of water in the tank changing at that time?

Solution:



Given: $\frac{dr}{dt} = -3$ when $h = 10$

Find: $\frac{dV}{dt}$ at that time

Here V represents the volume. V, r, and h all vary with time. We have:

$$V = \frac{\pi}{3} r^2 h$$

We next eliminate h from this equation. By similar triangles, $\frac{h}{r} = \frac{40}{8} = 5$. Thus $h = 5r$. Substituting this expression for h into the above equation, we get:

$$V = \frac{\pi}{3} r^2 (5r)$$

$$V = \frac{5\pi}{3} r^3$$

Taking $\frac{d}{dt}$ of both sides of equation:

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{5\pi}{3} r^3\right)$$

$$\frac{dV}{dt} = 5\pi r^2 \frac{dr}{dt}$$

We are given that $\frac{dr}{dt} = -3$. In addition, when $h = 10$, $r = 2$ (this follows from $h = 5r$, which we showed above). Substituting these values:

$$\frac{dV}{dt} = 5\pi(2)^2(-3) = -60\pi \text{ cm}^3 \text{ per second}$$

4. An object thrown into the air reaches a height of $h(t) = 32t - t^4$ feet after t seconds.

(a) (9 points) Find the maximum height the object reaches.

Solution: The velocity of the object at time t is $v(t) = h'(t) = 32 - 4t^3$. The object reaches its maximum height when its velocity $v(t)$ is 0:

$$32 - 4t^3 = 0$$

$$4t^3 = 32$$

$$t^3 = 8$$

$$t = 2$$

Thus the object reaches its maximum height at $t = 2$ seconds. At this time, its height is $h(2) = 32(2) - (2)^4 = 48$ ft.

(b) (6 points) When the object reaches its maximum height, what is its acceleration?

Solution: The acceleration of the object at time t is $a(t) = v'(t) = -12t^2$. At $t = 2$ seconds, the acceleration is $a(2) = -12(2)^2 = -48$ ft/sec^2.

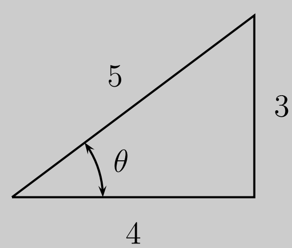
5. (4 points) This is a multiple choice problem. Circle the one correct answer. Let $f(x)$ and $g(x)$ be two functions which are differentiable for all x. Then $\frac{d}{dx}f(g(x)) =$

(a) $f'(g'(x))$
(b) $f'(g(x))$
(c) $f(g'(x))$
(d) $f'(g(x))g'(x)$
(e) $f'(g'(x))g'(x)$
(f) $f'(x)g'(x)$
(g) $f'(x)g(x) + f(x)g'(x)$
(h) $f'(x)g'(x) + f'(x)g'(x)$
(i) $g'(x)\ln(f(x))$

Solution: (d). Note that $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ is the chain rule.

6. (5 points) Find $\tan(\cos^{-1}(4/5))$. Your answer should not involve any trigonometric or inverse trigonometric functions. **Show your work**.

Solution: Consider the following right triangle:



Since $\cos(\theta) = 4/5$, it follows that $\theta = \cos^{-1}(4/5)$. Therefore

$$\tan(\cos^{-1}(4/5)) = \tan(\theta) = 3/4.$$