MATH 2250 Exam 3 Solutions

April 9, 2009

1. (12 points) Let $f(x)$ be a function, and suppose that $f(5) = 2$ and $f'(5) = 3$.

   (a) What is the linear approximation of $f(x)$ at $x = 5$?

   Solution: By the formula we learned in class, $L(x) = f(5) + f'(5)(x - 5)$. Substituting the given values for $f(5)$ and $f'(5)$, we have $L(x) = 2 + 3(x - 5)$.

   (b) Use (a) to approximate $f(5.3)$.

   Solution: Since 5.3 is close to 5, $f(5.3)$ can be approximated by $L(5.3)$, which equals $2 + 3(5.3 - 5) = 2.9$.

2. (15 points) For each of the following, answer True or False.

   (a) If $f(x)$ is continuous on $(-\infty, \infty)$, then $f(x)$ must have both an absolute maximum value and an absolute minimum value on $(-\infty, \infty)$.

   Solution: False. For example, the function $f(x) = x^2$ is continuous on $(-\infty, \infty)$ but has no absolute maximum value.

   (Note that if a function $f(x)$ is continuous on an interval of the form $[a, b]$, then it is true that $f(x)$ must have both an absolute maximum value and an absolute minimum value on $[a, b]$. This is the Extreme Value Theorem.)

   (b) If $f'(x)$ is increasing on an interval (a, b), then $f(x)$ is increasing on (a, b).

   Solution: False. For example, consider $f(x) = x^2$. The derivative $f'(x) = 2x$ is increasing on $(-1, 0)$, but $f(x)$ is decreasing on $(-1, 0)$.

   (c) If c is a critical point of $f(x)$, then $f(x)$ must either have a local maximum or a local minimum at $x = c$.

   Solution: False. For example, $x = 0$ is a critical point of the function $f(x) = x^3$, but the function has neither a local maximum nor a local minimum at $x = 0$.

   (d) If c is a critical point of $f(x)$, and $f''(c) < 0$, then $f(x)$ has a local maximum at $x = c$.

   Solution: True. This is the second derivative test. The idea behind the test is that if a function is concave down at a critical point, then the critical point must be a local maximum.

   (e) If $f'(x) < 0$ on an interval (a, b), then it is possible that $f(x)$ may be concave up on (a, b).

   Solution: True. For example, consider the function $f(x) = x^2$. Its derivative $f'(x) = 2x$ is less than 0 on $(-1, 0)$, and $f(x)$ is concave up on $(-1, 0)$.

(Basically (e) is asking whether it is possible for a function to be both decreasing and concave up on an interval (a, b).) Note: Since the exam had a typo in the statement of (e) (which has been corrected here), (e) was not graded.

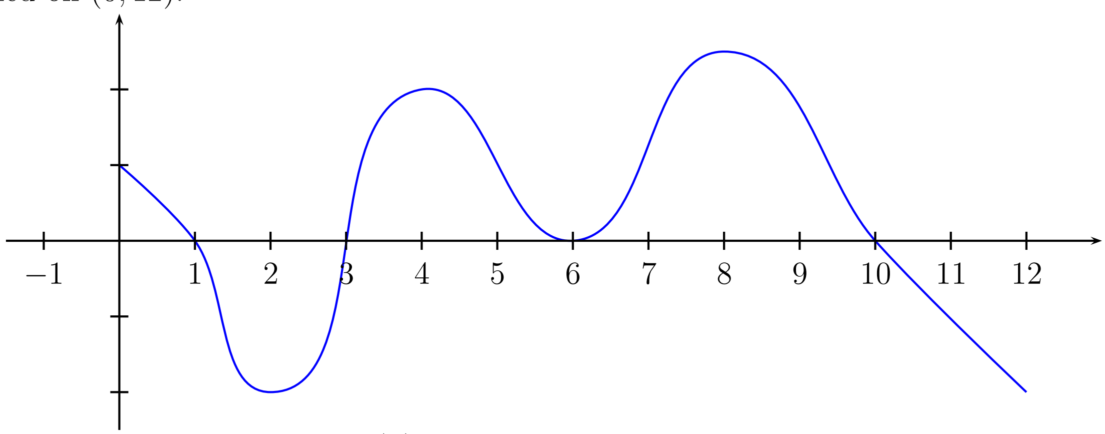
3. (13 points) Let $f(x) = \dfrac{x^2}{(x-3)^2}$. You may use the following facts:

$$f'(x) = \frac{-6x}{(x-3)^2} \quad \text{and} \quad f''(x) = \frac{12x+18}{(x-3)^4}.$$

Find all inflection points of $f(x)$, or state if there are none. Show all work supporting your assertions.

Solution: $f''(x) = 0$ when $x = -3/2$, and there are no other points in the domain of $f(x)$ where $f''(x) = 0$ or $f''(x)$ is undefined. Thus the only possible inflection point is at $x = -3/2$. One can check that $f''(x) < 0$ on $(-\infty, -3/2)$, and $f''(x) > 0$ on $(-3/2, \infty)$. Therefore $f(x)$ is concave down on $(-\infty, -3/2)$ and $f(x)$ is concave up on $(-3/2, \infty)$. Hence $f(x)$ changes concavity at $x = -3/2$, so $x = -3/2$ is an inflection point of $f(x)$.

4. (10 points) The figure below shows the graph of **the derivative** $f'(x)$ of a function defined on $(0, 12)$.



(a) Give all critical points of $f(x)$.

Solution: The critical points are the points x in the domain of $f(x)$ where $f'(x) = 0$ or $f'(x)$ is undefined. $f'(x)$ is never undefined on $(0, 12)$, and $f'(x) = 0$ at $x = 1$, $x = 3$, $x = 6$, and $x = 10$.

(b) For what values of x in $(0, 12)$ is $f(x)$ decreasing? Express your answer in interval notation.

Solution: $(1, 3) \cup (10, 12)$ (since these are the intervals where $f'(x) < 0$)

5. (24 points) Find the absolute maximum value and absolute minimum value of the function $f(x) = 3x^{2/3} - 2x$ on the interval $[-1, 8]$.

Solution: We first find the critical points of $f(x)$.

$$f'(x) = 3(2/3)x^{-1/3} - 2 = 2x^{-1/3} - 2 = (2/\sqrt[3]{x}) - 2$$

$\underline{f'(x) = 0}$

$(2/\sqrt[3]{x}) - 2 = 0$

$(2/\sqrt[3]{x}) = 2$

$\sqrt[3]{x} = 1$

$x = 1$

$\underline{f'(x) \text{ undefined}}$

$x = 0$

Thus $f(x)$ has two critical points: $x = 0$ and $x = 1$.

We next evaluate $f(x)$ at the critical points ($x = 0$ and $x = 1$) and the endpoints ($x = -1$ and $x = 8$):

$f(0) = 3(0)^{2/3} - 2(0) = 3(\sqrt[3]{0})^2 - 2(0) = 0 - 0 = 0$

$f(1) = 3(1)^{2/3} - 2(1) = 3(\sqrt[3]{1})^2 - 2(1) = 3 - 2 = 1$

$f(-1) = 3(-1)^{2/3} - 2(-1) = 3(\sqrt[3]{-1})^2 - 2(-1) = 3 + 2 = 5$

$f(8) = 3(8)^{2/3} - 2(8) = 3(\sqrt[3]{8})^2 - 2(8) = 12 - 16 = -4$

The function $f(x) = 3x^{2/3} - 2x$ is continuous on $[-1, 8]$. Hence, by the Extreme Value Theorem, $f(x)$ must have both an absolute maximum and an absolute minimum value on $[-1, 8]$. These values must occur at either critical points or endpoints. Therefore the absolute maximum value of $f(x)$ is 5, and it occurs at $x = -1$, and the absolute minimum value of $f(x)$ is -4, and it occurs at $x = 8$.

We are done. However, we show here a slightly different method to compute the critical points of $f(x)$.

$$f'(x) = 3(2/3)x^{-1/3} - 2 = 2x^{-1/3} - 2 = 2x^{-1/3}(1 - x^{1/3}) = 2\left(\frac{1 - x^{1/3}}{x^{1/3}}\right) = 2\left(\frac{1 - \sqrt[3]{x}}{\sqrt[3]{x}}\right)$$

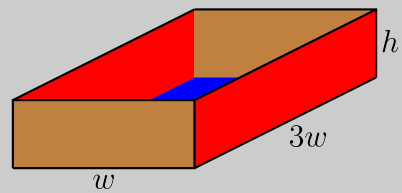
The critical points of $f(x)$ are the points where $f'(x) = 0$ or $f'(x)$ is undefined.

$f'(x) = 0$	$f'(x)$ undefined
$1 - \sqrt[3]{x} = 0$	$\sqrt[3]{x} = 0$
$1 = \sqrt[3]{x}$	$x = 0$
$x = 1$	

6. (26 points) A person wishes to construct a cardboard box with an open top and vertical sides, such that the length of the base is three times the width of the base. The surface area of the box is to be 36 ft^2 (since only 36 ft^2 of cardboard is available).

(a) What is the maximum possible volume of the box? What are the dimensions of the box of maximum volume?

Solution:



Maximize: $V = w \cdot 3w \cdot h$

Constraint:

Surface Area $= 36$

$2wh + 2(3w)h + w(3w) = 36$

$8wh + 3w^2 = 36$

$8wh = 36 - 3w^2$, so $h = (36 - 3w^2)/8w$

Substitute this expression for h back into the equation for V:

$$V(w) = w \cdot 3w \cdot ((36 - 3w^2)/8w) = (9/8)w(12 - w^2) = (9/8)(12w - w^3)$$

We wish to find the maximum value of $V(w)$, for w in $(0, \infty)$.

Critical points of $V(w)$:

$V'(w) = (9/8)(12 - 3w^2)$

$V'(w) = 0$ when $12 - 3w^2 = 0$, i.e., $w^2 = 4$, or $w = \pm 2$.

$V'(w)$ is never undefined.

Therefore $V(w)$ has exactly one critical point for w in $(0, \infty)$: $w = 2$. In (b), we will show that $V(w)$ has an absolute maximum at this critical point.

Answer:

Maximum volume $= V(2) = (9/8)(12(2) - 2^3) = (9/8)(16) = 18$ ft^3

Dimensions of box of maximum volume:

$w = 2$ ft, $l = 3w = 6$ ft, $h = (36 - 3w^2)/8w = 1.5$ ft

(b) Explain how you know the volume you found in (a) is the max possible volume.

Solution: Method 1. One shows that $V'(w) > 0$ on (0,2) and $V'(w) < 0$ on $(2, \infty)$. Hence $V(w)$ is increasing on $(0, 2)$ and decreasing on $(2, \infty)$. Therefore $V(w)$ has an absolute maximum at $w = 2$.

Method 2. $V''(w) = (9/8)(-6w) = (-27/4)w$, which is < 0 for all w in $(0, \infty)$. Hence $V(w)$ is concave down on all of $(0, \infty)$. Therefore $V(w)$ must have an absolute maximum at the critical point $w = 2$.