MATH 2250 Exam 4

April 28, 2009

1. (45 points) Evaluate each of the following. Do not simplify your answers.

(a) $\displaystyle\int \left(\frac{\sqrt{x} - x^2}{x^3} \right) dx.$

Solution:

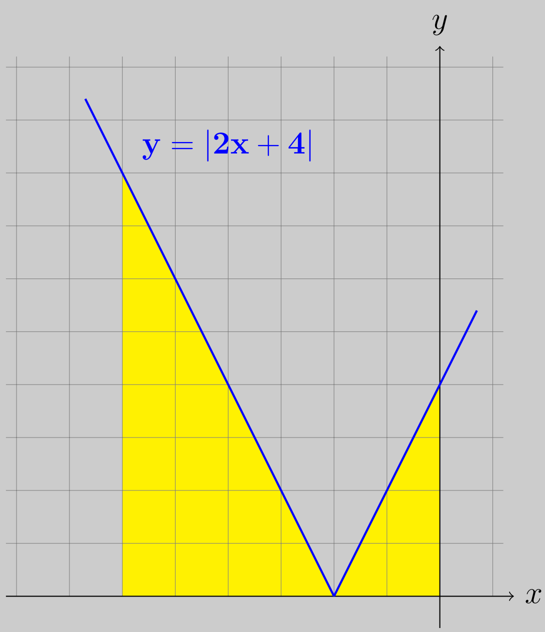
$$\int \left(\frac{\sqrt{x} - x^2}{x^3} \right) dx = \int \left(\frac{\sqrt{x}}{x^3} - \frac{x^2}{x^3} \right) dx = \int \left(\frac{x^{1/2}}{x^3} - \frac{x^2}{x^3} \right) dx$$
$$= \int \left(x^{-2.5} - x^{-1} \right) dx$$
$$= \frac{x^{-1.5}}{-1.5} - \ln|x| + C$$
$$= (-2/3)x^{-3/2} - \ln|x| + C$$

(b) $\displaystyle\int \left(\frac{-6}{\sqrt{1-x^2}} + 1 \right) dx.$

Solution: $\displaystyle\int \left(\frac{-6}{\sqrt{1-x^2}} + 1 \right) dx = -6\sin^{-1}(x) + x + C$

(c) $\displaystyle\int_{-6}^{0} |2x+4|\, dx$

Solution:



$$\int_{-6}^{0} |2x+4|\, dx = \text{ area of shaded region}$$
$$= \frac{(4)(8)}{2} + \frac{(2)(4)}{2}$$
$$= 16 + 4$$
$$= 20$$

(d) $\displaystyle\int_1^3 x^2(x^3-4)\,dx$

Solution:

$$\int_1^3 x^2(x^3-4)\,dx = \int_1^3 (x^5-4x^2) = \left(\frac{x^6}{6} - 4\,\frac{x^3}{3}\right)\Bigg|_1^3$$
$$= \left(\frac{3^6}{6} - 4\,\frac{3^3}{3}\right) - \left(\frac{1^6}{6} - 4\,\frac{1^3}{3}\right) = \frac{260}{3}$$

2. (9 points) True or False: $\displaystyle\int \frac{1}{x^2(x+1)}\,dx = \frac{-1}{x} + \ln(x+1) - \ln(x) + C.$
Justify your answer.

Solution: True.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{-1}{x} + \ln(x+1) - \ln(x)\right) &= \frac{d}{dx}\left(-x^{-1} + \ln(x+1) - \ln(x)\right) \\
&= x^{-2} + \frac{1}{x+1} - \frac{1}{x} \\
&= \frac{1}{x^2} + \frac{1}{x+1} - \frac{1}{x} \\
&= \frac{(x+1)}{x^2(x+1)} + \frac{x^2}{x^2(x+1)} - \frac{x(x+1)}{x^2(x+1)} \\
&= \frac{(x+1) + x^2 - x(x+1)}{x^2(x+1)} \\
&= \frac{x+1+x^2-x^2-x}{x^2(x+1)} = \frac{1}{x^2(x+1)}
\end{aligned}$$

Therefore $\dfrac{-1}{x} + \ln(x+1) - \ln(x)$ is an antiderivative of $\dfrac{1}{x^2(x+1)}$.

3. (7 points) Evaluate the sum $\displaystyle\sum_{k=0}^{3}((-1)^k(k-2)^2+2k)$.

Solution:

$$\begin{aligned}
\sum_{k=0}^{3}((-1)^k(k-2)^2+2k) &= ((-1)^0(0-2)^2+2(0)) + ((-1)^1(1-2)^2+2(1)) \\
&\quad + ((-1)^2(2-2)^2+2(2)) + ((-1)^3(3-2)^2+2(3)) \\
&= (2^2+0) + (-(-1)^2+2) + (0^2+4) + (-(1)^2+6) \\
&= 4+1+4+5 = 14
\end{aligned}$$

4. (16 points) Suppose that

$$\int_3^{12.5} f(x)\,dx = 6,\ \int_6^{12.5} f(x)\,dx = -2,\ \text{and}\ \int_3^9 f(x)\,dx = 4.$$

Find $\displaystyle\int_6^9 (5f(x) - 4x)\,dx$.

Solution:

$$\begin{aligned}
\int_6^9 (5f(x) - 4x)\,dx &= \int_6^9 5f(x)\,dx - \int_6^9 4x\,dx \\
&= 5\int_6^9 f(x)\,dx - \int_6^9 4x\,dx \\
&= 5\left[\int_6^{12.5} f(x)\,dx + \int_{12.5}^3 f(x)\,dx + \int_3^9 f(x)\,dx\right] - \int_6^9 4x\,dx \\
&= 5\left[\int_6^{12.5} f(x)\,dx - \int_3^{12.5} f(x)\,dx + \int_3^9 f(x)\,dx\right] - \int_6^9 4x\,dx \\
&= 5[-2 - 6 + 4] - \int_6^9 4x\,dx \\
&= 5[-2 - 6 + 4] - 4\left.\frac{x^2}{2}\right|_6^9 \\
&= 5[-2 - 6 + 4] - 4\left(\frac{9^2}{2} - \frac{6^2}{2}\right) \\
&= -20 - 90 = -110
\end{aligned}$$

5. (8 points) Suppose that $F(x) = \displaystyle\int_{-5}^{x^7} \frac{\sin t}{t^2 + t + 1}\,dt$. Find $F'(x)$.

Solution: By the Fundamental Theorem of Calculus and the Chain Rule,

$$\begin{aligned}
\frac{d}{dx}\left[\int_{-5}^{x^7} \frac{\sin t}{t^2 + t + 1}\,dt\right] &= \frac{\sin(x^7)}{(x^7)^2 + (x^7) + 1} \cdot (x^7)' \\
&= \frac{\sin(x^7)}{(x^7)^2 + (x^7) + 1} \cdot 7x^6
\end{aligned}$$

6. (15 points) Find $\lim_{x\to 0^+} x(\ln x)^2$. **Show all of your steps**.

Solution: Note that $\lim_{x\to 0^+} x(\ln x)^2$ is of type $0 \cdot -\infty$, an indeterminate form. This limit can be expressed as $\lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}}$, which is of type $\frac{\infty}{\infty}$. So l'Hôpital's Rule can be applied:

$$\begin{aligned}
\lim_{x\to 0^+} x(\ln x)^2 &= \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} && \left(\text{type } \frac{\infty}{\infty}\right) \\
&= \lim_{x\to 0^+} \frac{((\ln x)^2)'}{(\frac{1}{x})'} && \text{(by l'Hôpital's Rule)} \\
&= \lim_{x\to 0^+} \frac{2(\ln x)^1(\ln x)'}{(x^{-1})'} \\
&= \lim_{x\to 0^+} \frac{2(\ln x)(\frac{1}{x})}{-x^{-2}} \\
&= \lim_{x\to 0^+} \frac{-2(\ln x)(x^{-1})}{x^{-2}} \\
&= \lim_{x\to 0^+} -2(\ln x)x && (\text{type } \infty \cdot 0) \\
&= \lim_{x\to 0^+} \frac{-2(\ln x)}{\frac{1}{x}} && \left(\text{type } \frac{\infty}{\infty}\right) \\
&= \lim_{x\to 0^+} \frac{(-2(\ln x))'}{(\frac{1}{x})'} && \text{(by l'Hôpital's Rule)} \\
&= \lim_{x\to 0^+} \frac{(-2(\ln x))'}{(x^{-1})'} \\
&= \lim_{x\to 0^+} \frac{-2 \cdot \frac{1}{x}}{-x^{-2}} \\
&= \lim_{x\to 0^+} -2 \cdot \frac{1}{x} \cdot x^2 \\
&= \lim_{x\to 0^+} -2x \\
&= -2(0) \\
&= 0
\end{aligned}$$