EXACTNESS AND SPLITTING

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In abelian category, as we know, if B is splittable, $B = A \oplus C$, then clearly we have a short exact sequence,

$$0 \to A \to B \to C \to 0.$$

But exactness dosen't necesarily imply splitting, generalized from Hatcher's example, we have a short exact sequence

$$0 \to \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^{m+k}} \oplus \mathbb{Z}_{p^{n-k}} \to \mathbb{Z}_{p^n} \to 0,$$

where m, n, and k are positive integers, $k < n$, and p is a prime number. And we define the homomorphism

$$\varphi : \mathbb{Z}_{p^m} \to \mathbb{Z}_{p^{m+k}} \oplus \mathbb{Z}_{p^{n-k}}$$

by $\varphi(1) = (p^k, 1)$, and the homomorphism

$$\psi : \mathbb{Z}_{p^{m+k}} \oplus \mathbb{Z}_{p^{n-k}} \to \mathbb{Z}_{p^n}$$

by $\psi(p^k, 1) = 0$ and $\psi(1, 0) = 1$. However, $\mathbb{Z}_{p^{m+k}} \oplus \mathbb{Z}_{p^{n-k}}$ is not split as a direct sum of $\mathbb{Z}_{p^m}$ and $\mathbb{Z}_{p^n}$.

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