

GEODESIC FLOW AND ENERGY FUNCTIONAL ON RIEMANNIAN MANIFOLD

LOUIS YANG LIU

Let M be a n -dimensional Riemannian manifold with metric g , $\xi_x \in TM$, $c(t)$ be a curve in TM starting at $c(0) = \xi_x$ with initial velocity $c'(0) = (X, \Xi) \in T_{\xi_x}TM$, then

$$(X, \Xi) \in H_{\xi_x}TM \Leftrightarrow \nabla_{\xi_x} X = 0$$

Writing in terms of coordinates $(x^1, \dots, x^n, \xi^1, \dots, \xi^n)$, $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x^i}$, and $\Xi = \sum_{i=1}^n \Xi_i \frac{\partial}{\partial \xi^i}$, and then we have

$$(0.1) \quad \Xi_k + \sum_{i,j=1}^n \Gamma_{ij}^k X_i \xi_j = 0$$

by $\nabla_{\xi_x} X = 0$.

On the other hand, for vertical space, we have

$$(X, \Xi) \in V_{\xi_x}TM \Leftrightarrow X = 0 \Leftrightarrow X_i = 0$$

for all i .

Here is a direct way to see that a geodesic flow is a Hamiltonian flow of energy functional.

In terms of coordinates, we know that for geodesic vector field

$$\mathcal{X}(\xi_x) := \sum_{k=1}^n \xi_k \frac{\partial}{\partial x^k} - \sum_{k=1}^n \sum_{i,j=1}^n \Gamma_{ij}^k \xi_i \xi_j \frac{\partial}{\partial \xi^k}$$

on TM by (0.1), we have

$$\begin{aligned} i_{\mathcal{X}(\xi_x)} d\alpha_{\xi_x}(\cdot) &= d\alpha_{(\xi_x)}(\mathcal{X}(\xi_x), \cdot) \\ &= dg(\xi_x, \pi_* (\sum_{k=1}^n \xi_k \frac{\partial}{\partial x^k} - \sum_{k=1}^n \sum_{i,j=1}^n \Gamma_{ij}^k \xi_i \xi_j \frac{\partial}{\partial \xi^k}))(\cdot) \\ &= dg(\xi_x, \xi_x)(\cdot) \\ &= dE_{\xi_x}(\cdot). \end{aligned}$$

Thus, let $\omega := d\alpha$ be the natural symplectic form on TM , we then have the following proposition.

Proposition 0.1. $i_{\mathcal{X}}\omega = dE$.

Date: April 23, 2008.

There are some relationship between the energy functional and natural differential forms on Riemannian manifold. We first consider the case $\Gamma_{ij}^k = 0$, it means $\Xi_k = 0$ for all k by (1) if $(X, \Xi) \in H_{\xi_x} TM$. There is a natural involution between horizontal space and vertical space which is

$$\begin{aligned} \iota : H_{\xi_x} TM &\rightarrow V_{\xi_x} TM \\ \iota(\sum k_i \frac{\partial}{\partial x^i} + 0 \frac{\partial}{\partial \xi^i}) &= \sum 0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i}. \end{aligned}$$

Thus

$$\begin{aligned} \alpha_{\xi_x} \circ \iota(\sum_{i=1}^n (0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i})) &= \alpha_{\xi_x}(\sum_{i=1}^n k_i \frac{\partial}{\partial x^i} + 0 \frac{\partial}{\partial \xi^i}) \\ &= g(\xi_x, \pi_*(\sum_{i=1}^n (k_i \frac{\partial}{\partial x^i} + 0 \frac{\partial}{\partial \xi^i}))) \\ &= \sum_{i,j=1}^n g_{ij} \xi_i k_j. \end{aligned}$$

On the other hand for dE , we have

$$\begin{aligned} dE_{\xi_x}(\sum_{i=1}^n 0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i}) &= \sum_{i=1}^n k_i \frac{\partial}{\partial \xi^i} E(\xi_x) \\ &= \sum_{i=1}^n k_i \frac{\partial}{\partial \xi^i} g(\xi_x, \xi_x) \\ &= \sum_{i,j=1}^n g_{ij} \xi_i k_j. \end{aligned}$$

Therefore

$$dE_{\xi_x}((0, \Xi)) = \alpha_{\xi_x} \circ \iota((0, \Xi))$$

for any $(0, \Xi) \in V_{\xi_x} TM$.

Now let us consider the general case. For any $(X, \Xi) \in H_{\xi_x} TM$, we have $\Xi_k = -\sum_{i,j=1}^n \Gamma_{ij}^k X_i \xi_j$, analogous to the case $\Gamma_{ij}^k = 0$, an involution we can have is

$$\begin{aligned} \iota : V_{\xi_x} TM &\rightarrow H_{\xi_x} TM \\ \iota(\sum 0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i}) &= \sum k_i \frac{\partial}{\partial x^i} - \sum \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_{\xi_x} \circ \iota(\sum_{i=1}^n (0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i})) &= \alpha_{\xi_x}(\sum_{i=1}^n k_i \frac{\partial}{\partial x^i} - \sum_{k=1}^n \sum_{i,j=1}^n \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k}) \\ &= g(\xi_x, \pi_*(\sum_{i=1}^n k_i \frac{\partial}{\partial x^i} - \sum_{k=1}^n \sum_{i,j=1}^n \Gamma_{ij}^k k_i \xi_j \frac{\partial}{\partial \xi^k})) \\ &= \sum_{i,j=1}^n g_{ij} \xi_i k_j. \end{aligned}$$

and

$$\begin{aligned}
 dE_{\xi_x} \left(\sum_{i=1}^n 0 \frac{\partial}{\partial x^i} + k_i \frac{\partial}{\partial \xi^i} \right) &= \sum_{i=1}^n k_i \frac{\partial}{\partial \xi^i} E(\xi) \\
 &= \sum_{i=1}^n k_i \frac{\partial}{\partial \xi^i} g(\xi, \xi) \\
 &= \sum_{i,j=1}^n g_{ij} \xi_i k_j.
 \end{aligned}$$

Thus we have proved the following theorem.

Theorem 0.2. *$dE = \alpha \circ \iota$ on the vertical bundle of TTM , where E is the energy functional on Riemannian manifold M and α is the natural 1-form on TM whose differential is the natural symplectic form on TM .*

Remark 0.3. In fact, since both sides are zeroes on the horizontal bundle of TTM , therefore $dE = \alpha \circ \iota$ on TTM .

Acknowledgement. Thanks to Dr. Joseph H. G. Fu's guidance.