The Inverse of Cosine Transform for Minkowski Metrics

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February 1, 2009

The goal of the expository notes is to find explicit Crofton measures for Minkowski metrics. The cosine transform, as one knows, is defined as

$$\mathcal{C}(g)(\cdot) = \int_{\xi \in S^1} |\langle \cdot, \xi \rangle| g(\xi)\, d\sigma(\xi), \tag{1}$$

where σ is the standard measure on S^1. An important theorem on cosine transform from (G), Proposition 3.6.4, states that

Theorem 1. *(Surjectivity of Cosine Transform) If f is a 4-times continuously differentiable even function on S^1 there exists some even continuous function g on S^1 such that $f = \mathcal{C}(g)$ on S^1.*

As we know, the Fourier coefficients of $f(\cos\theta, \sin\theta)$ are

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(\cos\theta, \sin\theta) \cos(n\theta)\, d\theta, \quad n \geq 0, \tag{2}$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(\cos\theta, \sin\theta) \sin(n\theta)\, d\theta, \quad n \geq 0, \tag{3}$$

in particular, $a_{2k+1} = b_{2k+1} = 0$ for $0 \leq k < \infty$ because F is an even function on S^1. Thus from the theory of Fourier series we know

$$f(\cos\theta, \sin\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_{2k} \cos 2k\theta + b_{2k} \sin 2k\theta). \tag{4}$$

From the proof to Theorem 1 in (G) one can deduce the formula for the inverse of cosine transform (1), which is

$$g(\theta) = \frac{1}{\kappa_1}\left(\frac{a_0}{2\lambda_{2,0}} + \sum_{k=1}^{\infty} \frac{1}{\lambda_{2,2k}}(a_{2k}\cos 2k\theta + b_{2k}\sin 2k\theta)\right), \tag{5}$$

where $\kappa_1 = 2$ is the volume of the one dimensional ball and

$$\lambda_{2,2k} = \frac{2(-1)^{k-1}}{(2k+1)(2k-1)} \tag{6}$$

for $0 \le k < \infty$.

For instance, suppose that one has an elliptic metric

$$F(x,y) = \sqrt{x^2 + 2y^2} \tag{7}$$

which is particularly a Minkowski metric. Let $f := F^2$, then

$$f(\cos\theta, \sin\theta) = \cos^2\theta + 2\sin^2\theta = \frac{3}{2} - \frac{1}{2}\cos 2\theta, \tag{8}$$

from which we know $a_0 = \frac{3}{2}$, $a_2 = -\frac{1}{2}$, and $a_{2k} = 0$ for $k > 1$ and $b_{2k} = 0$ for for $0 \le k < \infty$. It follows from (5) that $g(\theta) = \frac{3}{8}(1 - \cos 2\theta)$. In other words,

$$\frac{3}{8}\int_0^{2\pi} |x\cos\theta + y\sin\theta|(1 - \cos 2\theta)\, d\theta = x^2 + 2y^2 \tag{9}$$

for any $(x,y) \in S^1$.

L^p spaces, $1 < p < \infty$, are interesting models of studying Minkowski spaces for various reasons. In fact, we have seen that the problem of finding explicit Crofton measure for a Minkowski metric is to obtain its Fourier series (4).

However, it turns out to be a large computation work for one to write the coefficients out explicitly for the general L^p metric. For instance, if one computes

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} (\cos^p\theta + \sin^p\theta)\, d\theta \tag{10}$$

by Mathematica 7.0, it gives us

$$a_0 = \frac{2^{1+p}(3(-1)^p + 1)\pi^2}{\pi\,(1 + (-1)^p)\,\Gamma(\frac{1-p}{2})^2\Gamma(1+p)}. \tag{11}$$

One can see the inputs and outputs of some computations in Mathematica 7.0

in the Appendix.

But we would like to provide one more nice example, L^4 space with metric $||(x,y)||_{L^4} = (x^4+y^4)^{1/4}$. The Fourier series for $f(\cos\theta, \sin\theta) = ||(\cos\theta, \sin\theta)||_{L^4}^4$ is

$$f(\cos\theta, \sin\theta) = \cos^4\theta + \sin^4\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta. \tag{12}$$

Plugging $\lambda_{2,0} = 2$ and $\lambda_{2,4} = -\frac{2}{15}$ into (5), we have the inverse of the cosine transform $g(\theta) = \frac{3}{16}(1 - 5\cos 4\theta)$ for L^4 metric.

Remark 2. We can draw recurrence relations between Fourier coefficients for $p = 2(m+1)$ and the ones for $p = 2m$ from

$$\cos^{2(m+1)}\theta = \frac{1}{2}(1+\cos 2\theta)\cos^{2m}\theta. \tag{13}$$

Let $\frac{a_0^{(m)}}{2} + \sum_{k=1}^{\infty} a_{2k}^{(m)} \cos 2k\theta$ be the Fourier series of $\cos^{2m}\theta$. Therefore

$$\begin{aligned} \tfrac{1}{2}(1+\cos 2\theta)\cos^{2m}\theta &= \tfrac{1}{2}\big(\tfrac{a_0^{(m)}}{2} + \tfrac{a_0^{(m)}}{2}\cos 2\theta + \textstyle\sum_{k=1}^{\infty} a_{2k}^{(m)} \cos 2k\theta \\ &\qquad + \textstyle\sum_{k=1}^{\infty} a_{2k}^{(m)} \cos 2\theta \cos 2k\theta\big) \\ &= \tfrac{1}{2}\big(\tfrac{a_0^{(m)}}{2} + \tfrac{a_0^{(m)}}{2}\cos 2\theta + \textstyle\sum_{k=1}^{\infty} a_{2k}^{(m)} \cos 2k\theta \\ &\qquad \tfrac{1}{2}\textstyle\sum_{k=1}^{\infty} a_{2k}^{(m)} (\cos 2(k+1)\theta + \cos 2(k-1)\theta)\big) \\ &= \tfrac{a_0^{(m)} + a_2^{(m)}}{4} + \textstyle\sum_{k=1}^{\infty} \tfrac{a_{2(k-1)}^{(m)} + 2a_{2k}^{(m)} + a_{2(k+1)}^{(m)}}{4} \cos 2k\theta. \end{aligned} \tag{14}$$

It follows that

$$a_0^{(m+1)} = \frac{a_0^{(m)} + a_2^{(m)}}{2} \tag{15}$$

and

$$a_{2k}^{(m+1)} = \frac{a_{2(k-1)}^{(m)} + 2a_{2k}^{(m)} + a_{2(k+1)}^{(m)}}{4} \tag{16}$$

for $k \geq 1$, and likewise for $\sin^{2(m+1)}\theta$. So one can obtain the Fourier series in L^{2m} for integer $m > 0$ and so the inverses of the cosine transforms.

Appendix

The following inputs and outputs are carried out in Mathematica 7.0, in which the lines in black are inputs and the others are outputs that however stagnate at computing the Fourier coefficient of $\cos 4\theta$.

$\boldsymbol{p > 1}$

$p > 1$

$$\mathbf{(1/\pi)\int_0^{2\pi}(Sin[x]^p + Cos[x]^p)\ dx}$$

$$\text{If}\left[\text{Re}[p] > -1, \frac{2^{1+p}e^{ip\pi}\left(3+e^{ip\pi}\right)\pi^2}{\pi(1+e^{ip\pi})\text{Gamma}\left[\frac{1}{2}-\frac{p}{2}\right]^2\text{Gamma}[1+p]}, \frac{1}{\pi}\text{Integrate}[\text{Cos}[\text{x}]^p+\text{Sin}[\text{x}]^p, \{x, 0, 2\pi\}, \text{Assumptions} \to \text{Re}[p] \leq -1]\right]$$

$$\mathbf{(1/\pi)\int_0^{2\pi}(Sin[x]^p + Cos[x]^p)\, Cos[2x]dx}$$

$$\frac{\text{If}[\text{Re}[p]>-1,0,\text{Integrate}[\text{Cos}[x]^p\text{Cos}[2x]+\text{Cos}[2x]\text{Sin}[x]^p,\{x,0,2\pi\},\text{Assumptions}\to\text{Re}[p]\leq-1]]}{\pi}$$

$$\mathbf{(1/\pi)\int_0^{2\pi}(Sin[x]^p + Cos[x]^p)\, Cos[4x]dx}$$

References

[G] H. Groemer, Geometric Applications of Fourier Series and Spherical Harmonics, Cambridge University Press, 1996.