KÄHLER FORM OF COMPLEX L^p SPACE AND LAGRANGIANS

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Abstract. We obtain the Kähler forms for complex L^p spaces, $1 \le p < \infty$, and all the lagrangians in this article. The symplectic structures determined by the Kähler forms help us to understand the support of the complex HT valuations in the integral geometry of complex L^p spaces.

1. Kähler Forms and Preliminaries

To obtain the canonical Kähler form for complex L^p space, which as one knows has the complex L^p norm

$$F_p(z,w) = ||(z,w)||_p := (|z|^p + |w|^p)^{1/p} \tag{1.1}$$

for $1 \le p < \infty$, we can use the generalized Kähler potential $G_p := F_p^2$, and then the Kähler form κ_p will be the negative imaginary part of $\frac{1}{2}\partial\overline{\partial}(G_p)$.

First, we have

$$\begin{aligned} \overline{\partial}(F_p^2) &= \tfrac{2}{p}(|z|^p + |w|^p)^{\frac{2}{p}-1}\overline{\partial}(|z|^p + |w|^p) \\ &= \tfrac{2}{p}(|z|^p + |w|^p)^{\frac{2}{p}-1}\overline{\partial}((z\overline{z})^{p/2} + (w\overline{w})^{p/2}) \\ &= (|z|^p + |w|^p)^{\frac{2}{p}-1}(|z|^{p-2}zd\overline{z} + |w|^{p-2}wd\overline{w}) \\ &= (1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1}zd\overline{z} + (1 + |\tfrac{z}{w}|^p)^{\frac{2}{p}-1}wd\overline{w}, \end{aligned} \tag{1.2}$$

in other word, $\frac{\partial}{\partial\overline{z}}F_p^2 = (1 + |\frac{w}{z}|^p)^{\frac{2}{p}-1}z$ and $\frac{\partial}{\partial\overline{w}}F_p^2 = (1 + |\frac{z}{w}|^p)^{\frac{2}{p}-1}w$.

Using $\frac{\partial}{\partial z}(|\frac{1}{z}|^p) = \frac{\partial}{\partial z}((z\overline{z})^{-\frac{p}{2}}) = (-\frac{p}{2})(z\overline{z})^{-\frac{p}{2}-1}\overline{z} = (-\frac{p}{2})\frac{1}{z|z|^p}$, we get

$$\begin{aligned} \tfrac{\partial}{\partial z}((1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1}z) &= (1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1} + z\tfrac{\partial}{\partial z}(1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1} \\ &= (1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1} + (\tfrac{2}{p} - 1)z(1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|w|^p\tfrac{\partial}{\partial z}(|\tfrac{1}{z}|^p) \\ &= (1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-1} + (\tfrac{p}{2} - 1)(1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p \\ &= (1 + |\tfrac{w}{z}|^p)^{\frac{2}{p}-2}(1 + \tfrac{p}{2}|\tfrac{w}{z}|^p), \end{aligned} \tag{1.3}$$

thus

$$\frac{\partial^2}{\partial z\partial\overline{z}} = (1 + |\frac{w}{z}|^p)^{\frac{2}{p}-2}(1 + \frac{p}{2}|\frac{w}{z}|^p). \tag{1.4}$$

and by symmetry of w and z in (1.4) we know

$$\frac{\partial^2}{\partial w\partial\overline{w}} = (1 + |\frac{z}{w}|^p)^{\frac{2}{p}-2}(1 + \frac{p}{2}|\frac{z}{w}|^p). \tag{1.5}$$

Date: December 18, 2008.

By $\frac{\partial}{\partial w}(|w|^p) = \frac{\partial}{\partial w}(w\overline{w})^{\frac{p}{2}} = \frac{p}{2}(w\overline{w})^{\frac{p}{2}-1}\overline{w} = \frac{p}{2}\frac{|w|^p}{w}$,

$$\begin{aligned} \frac{\partial}{\partial w}((1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-1}z) &= (\tfrac{2}{p}-1)z(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}\tfrac{1}{|z|^p}\tfrac{p}{2}\tfrac{|w|^p}{w} \\ &= (1-\tfrac{p}{2})\tfrac{z}{w}|\tfrac{w}{z}|^p(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}, \end{aligned} \tag{1.6}$$

thus

$$\frac{\partial^2}{\partial w\partial \overline{z}}F_p^2 = (1-\frac{p}{2})\frac{z}{w}|\frac{w}{z}|^p(1+|\frac{w}{z}|^p)^{\frac{2}{p}-2}. \tag{1.7}$$

By symmetry of z and w in (1.7), we also have

$$\frac{\partial^2}{\partial z\partial \overline{w}}F_p^2 = (1-\frac{p}{2})\frac{w}{z}|\frac{z}{w}|^p(1+|\frac{z}{w}|^p)^{\frac{2}{p}-2}. \tag{1.8}$$

So we have the following

Theorem 1.1. *The Hermitian metric for complex L^p plane induced from its norm is*

$$\begin{aligned} h_p = \ & \tfrac{1}{2}(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}((1+\tfrac{p}{2}|\tfrac{w}{z}|^p)dz\otimes d\overline{z} + (1-\tfrac{p}{2})\tfrac{z}{w}|\tfrac{w}{z}|^p dw\otimes d\overline{z}) \\ & +\tfrac{1}{2}(1+|\tfrac{z}{w}|^p)^{\frac{2}{p}-2}((1-\tfrac{p}{2})\tfrac{w}{z}|\tfrac{z}{w}|^p dz\otimes d\overline{w} + (1+\tfrac{p}{2}|\tfrac{z}{w}|^p)dw\otimes d\overline{w}). \end{aligned} \tag{1.9}$$

Let $z = x+yi$ and $w = u+vi$, then we can express the Kähler form by taking the negative imaginary part of (1.9), which is the following

Corollary 1.2. *The canonical Kähler form κ_p for complex L^p plane induced from its norm is*

$$\begin{aligned} \kappa_p = \ & (1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{w}{z}|^p)dx\wedge dy \\ & +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Re(\tfrac{z}{w})(dx\wedge dv - dy\wedge du) \\ & +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Im(\tfrac{z}{w})(dx\wedge du + dy\wedge dv) \\ & +(1+|\tfrac{z}{w}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{z}{w}|^p)du\wedge dv. \end{aligned} \tag{1.10}$$

One can also take the real part of (1.9), which is a Riemannian metric denoted by g_p. The Kähler form κ_p combines the Riemannian metric g_p and the complex structure J as follows

Corollary 1.3. *Given any $(z_1, w_1), (z_2, w_2)$ in complex L^p plane, then*

$$g_p((z_1,w_1),(z_2,w_2)) = \kappa_p((z_1,w_1),J(z_2,w_2)). \tag{1.11}$$

Proof. First, the real part of (1.9) yields

$$\begin{aligned} g_p = \ & (1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{w}{z}|^p)(dx\otimes dx + dy\otimes dy) \\ & +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Re(\tfrac{z}{w})((du\otimes dx + dv\otimes dy) \\ & \qquad +(dx\otimes du + dy\otimes dv)) \\ & +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Im(\tfrac{z}{w})((du\otimes dy - dv\otimes dx) \\ & \qquad +(dy\otimes du - dx\otimes dv)) \\ & +(1+|\tfrac{z}{w}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{z}{w}|^p)(du\otimes du + dv\otimes dv). \end{aligned} \tag{1.12}$$

We have the following equalities,

$$(dx \otimes dx + dy \otimes dy)((z_1, w_1), (z_2, w_2)) = Re(z_1 \overline{z_2}) = dx \wedge dy((z_1, w_1), J(z_2, w_2)) \tag{1.13}$$

and

$$\begin{aligned} & ((du \otimes dx + dv \otimes dy) + (dx \otimes du + dy \otimes dv))((z_1, w_1), (z_2, w_2)) \\ = \; & Re(z_1 \overline{w_2} + z_2 \overline{w_1}) \\ = \; & (dx \wedge dv - dy \wedge du)((z_1, w_1), J(z_2, w_2)), \end{aligned} \tag{1.14}$$

similarly,

$$(du \otimes du + dv \otimes dv)((z_1, w_1), (z_2, w_2)) = Re(w_1 \overline{w_2}) = du \wedge dv((z_1, w_1), J(z_2, w_2)) \tag{1.15}$$

and

$$\begin{aligned} & ((du \otimes dy - dv \otimes dx) + (dy \otimes du - dx \otimes dv))((z_1, w_1), (z_2, w_2)) \\ = \; & Im(z_1 \overline{w_2} + z_2 \overline{w_1}) \\ = \; & (dx \wedge du + dy \wedge dv)((z_1, w_1), J(z_2, w_2)). \end{aligned} \tag{1.16}$$

Then comparing (1.10) and (1.12), (1.11) in the claim follows. □

Our goal is to get all Lagrangian subspaces of the Kähler form κ_p for every $1 < p < \infty$. For clearness, we consider the case of complex L^2 and L^1 spaces first, and then generalize them. Since

$$\kappa_2 = dx \wedge dy + du \wedge dv \tag{1.17}$$

from (1.12), we have

Theorem 1.4. *The set of Lagrangian subspaces of* $\mathbb{C}^2$ *with complex* L^2 *norm is*

$$\widetilde{\mathbf{T}} := \{span((z_1, w_1), (z_2, w_2)) : z_1, w_1, z_2, w_2 \in \mathbb{C}, Im(z_2 \overline{z_1}) = Im(w_1 \overline{w_2})\}. \tag{1.18}$$

Proof. Suppose κ_2 vanishes on a plane P spanned by (z_1, w_1) and (z_2, w_2). Since

$$dx \wedge dy((z_1, w_1), (z_2, w_2)) = Im(z_2 \overline{z_1}) \tag{1.19}$$

and

$$du \wedge dv((z_1, w_1), (z_2, w_2)) = Im(w_2 \overline{w_1}) \tag{1.20}$$

we then have

$$Im(z_2 \overline{z_1}) + Im(w_2 \overline{w_1}) = 0, \tag{1.21}$$

i.e. $Im(z_2 \overline{z_1}) = Im(w_1 \overline{w_2})$. □

Remark 1.5. Obviously $L \notin \mathbf{T}_0 \cup \widetilde{\mathbf{T}}$ for any complex line L in $\mathbb{C}^2$. Moreover, one can show that $\mathbf{T}_0 \cup \widetilde{\mathbf{T}} = \{\varphi(span((1,0),(0,1))) : \varphi \in U(2)\}$ by normalizing the basis $\{(z_1, w_1), (z_2, w_2)\}$, see [T].

However, we have

$$\begin{aligned}\kappa_1 &= (1+\tfrac{1}{2}|\tfrac{w}{z}|)dx\wedge dy + (1+\tfrac{1}{2}|\tfrac{z}{w}|)du\wedge dv \\ &\quad +\tfrac{1}{2}Re(\tfrac{z}{w}|\tfrac{w}{z}|)(dx\wedge dv - dy\wedge du) + \tfrac{1}{2}Im(\tfrac{z}{w}|\tfrac{w}{z}|)(dx\wedge du + dy\wedge dv).\end{aligned} \tag{1.22}$$

Using the equalities (1.19), (1.20),

$$(dx\wedge dv - dy\wedge du)((z_1,w_1),(z_2,w_2)) = Im(w_2\overline{z_1} - w_1\overline{z_2}) \tag{1.23}$$

and

$$(dx\wedge du + dy\wedge dv)((z_1,w_1),(z_2,w_2)) = Re(w_2\overline{z_1} - w_1\overline{z_2}), \tag{1.24}$$

we get

$$\begin{aligned}\kappa_1((z_1,w_1),(z_2,w_2)) &= (1+\tfrac{1}{2}|\tfrac{w}{z}|)Im(z_2\overline{z_1}) + (1+\tfrac{1}{2}|\tfrac{z}{w}|)Im(w_2\overline{w_1}) \\ &\quad +\tfrac{1}{2}Im(\tfrac{z}{w}|\tfrac{w}{z}|(w_2\overline{z_1} - w_1\overline{z_2})).\end{aligned} \tag{1.25}$$

Now let's state the result of Lagrangians for L^1 and prove it by analyzing several cases.

Theorem 1.6. *The set of Lagrangian subspaces of* $\mathbb{C}^2$ *with complex* L^1 *norm is* $\mathbb{T}^{\mathbf{2}}\cup\mathbb{T}^1$, *where*

$$\mathbb{T}^{\mathbf{2}} := \{span((z,0),(0,w)) : z,w\in U(1)\} \cong U(1)\times U(1) \tag{1.26}$$

and

$$\mathbb{T}^1 := \{P : P = \{\lambda(z,w) : \lambda\in\mathbb{R}, z,w\in U(1),\ zw \text{ is a constant in } U(1)\}\} \cong U(1). \tag{1.27}$$

Proof. Firstly we can show that

$$P = \{\lambda(z,w) : \lambda\in\mathbb{R}, z,w\in U(1),\ zw \text{ is a constant in } U(1)\} \tag{1.28}$$

is identical to some

$$P' := span((z_1, z_1e^{i\theta}),(z_2, \frac{z_1^2\overline{z_2}}{|z_1|^2}e^{i\theta})) \tag{1.29}$$

where $z_1,z_2\in\mathbb{C}\setminus\{0\}$. For any $\lambda(e^{i\varphi},e^{i\psi})\in P$, let $z_1=\lambda e^{i\varphi}$, $\theta=\psi-\varphi$, we have $P = span((z_1,z_1e^{i\theta}),(z_2,\frac{z_1^2\overline{z_2}}{|z_1|^2}e^{i\theta})) = P'$ where $z_2\in\mathbb{C}\setminus\{0\}$.

It is not hard to see $\kappa_1(z_1,0),(0,z_2)) = 0$ in (1.25). On the other hand, for any

$$(z,w) = \lambda_1(z_1,z_1) + \lambda_2(z_2,\frac{z_1^2\overline{z_2}}{|z_1|^2}) \in span((z_1,z_1),(z_2,\frac{z_1^2\overline{z_2}}{|z_1|^2})), \tag{1.30}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned} |w|^2 &= (\lambda_1 z_1 + \lambda_2 \tfrac{z_1^2 \overline{z_2}}{|z_1|^2})(\lambda_1 \overline{z_1} + \lambda_2 \tfrac{\overline{z_1}^2 z_2}{|z_1|^2}) \\ &= \lambda_1^2 z_1 \overline{z_1} + \lambda_1 \lambda_2 \overline{z_1} z_2 + \lambda_2 \lambda_1 z_1 \overline{z_2} + \lambda_2^2 \overline{z_2} z_2 \\ &= (\lambda_1 z_1 + \lambda_2 z_2)(\lambda_1 \overline{z_1} + \lambda_2 \overline{z_2}) \\ &= |z|^2, \end{aligned} \tag{1.31}$$

that implies $|\frac{w}{z}| = 1$. Therefore we have

$$\begin{aligned} \kappa_{(z,w)}((z_1, z_1), (z_2, \tfrac{z_1^2 \overline{z_2}}{|z_1|^2})) &= \tfrac{3}{2}(Im(z_2 \overline{z_1}) + \tfrac{3}{2} Im(\tfrac{z_1^2 \overline{z_2}}{|z_1|^2} \overline{z_1})) \\ &\quad - \tfrac{1}{2} Im(\tfrac{z}{w} |\tfrac{w}{z}| (\tfrac{z_1^2 \overline{z_2}}{|z_1|^2} \overline{z_1} - z_1 \overline{z_2})) \\ &= \tfrac{3}{2}(Im(z_2 \overline{z_1}) + Im(z_1 \overline{z_2})) \\ &= 0. \end{aligned} \tag{1.32}$$

So κ vanishes on $span((z_1, z_1), (z_2, \frac{z_1^2 \overline{z_2}}{|z_1|^2}))$ for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, $Im(\frac{z_1}{z_1}) \neq 0$.

Conversely, suppose that κ vanishes on a plane P spanned by (z_1, w_1) and (z_2, w_2). From (1.25), we know that

$$(1+\frac{1}{2}|\frac{w}{z}|)Im(z_2\overline{z_1})+(1+\frac{1}{2}|\frac{z}{w}|)Im(w_2\overline{w_1})+\frac{1}{2}Im(\frac{z}{w}|\frac{w}{z}|(w_2\overline{z_1}-w_1\overline{z_2}))=0 \tag{1.33}$$

holds for any $(z, w) \in span((z_1, w_1), (z_2, w_2))$. In the following argument, we divide it into three cases to discuss in terms of $|\frac{w}{z}|$ and $\frac{z}{w}|\frac{w}{z}|$.

The first case is that $|\frac{w}{z}| = \lambda$ for some fixed $\lambda > 0$. let $(z, w) = \lambda_1(z_1, w_1) + \lambda_2(z_2, w_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$, then $|\lambda_1 w_1 + \lambda_2 w_2| = \lambda |\lambda_1 z_1 + \lambda_2 z_2|$, that implies $|w_1| = \lambda |z_1|$, $|w_2| = \lambda |z_2|$ and $Re(w_1 \overline{w_2}) = \lambda^2 Re(z_1 \overline{z_2})$. It follows that either $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$, or $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} \frac{z_1^2 \overline{z_2}}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$.

In the sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$ for some $\theta \in [0, 2\pi)$, by (1.33) we have

$$(1+\frac{\lambda}{2})Im(z_2\overline{z_1})+(1+\frac{1}{2\lambda})\lambda^2 Im(z_2\overline{z_1})+\lambda Im(z_2\overline{z_1}) = (1+\lambda)^2 Im(z_2\overline{z_1}) = 0 \tag{1.34}$$

which gives $Im(z_2 \overline{z_1}) = 0$, and then $Im(w_2 \overline{w_1}) = 0$, that means (z_1, w_1) and (z_2, w_2) are colinear. So this case is not allowed.

However, for the other sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} \frac{z_1^2 \overline{z_2}}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$, by (1.33) we have

$$(1 + \frac{\lambda}{2})Im(z_2\overline{z_1}) + (1 + \frac{1}{2\lambda})\lambda^2 Im(z_1\overline{z_2}) = (1 - \lambda^2)Im(z_2\overline{z_1}) = 0. \tag{1.35}$$

Then $\lambda = 1$ or $Im(z_2 \overline{z_1}) = 0$, but (z_1, w_1) and (z_2, w_2) can not be colinear. So we have $\lambda = 1$ which gives

$$P = span((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1^2 \overline{z_2}}{|z_1|^2} e^{i\theta})), \tag{1.36}$$

where $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $Im(z_1 \overline{z_2}) \neq 0$ for some $\theta \in [0, 2\pi)$, that finishes the first case.

The second case is $\frac{w}{z}|\frac{w}{z}| = e^{i\theta}$ for some fixed $\theta \in [0, 2\pi)$. Let $w_1 = \lambda_1 e^{i\theta} z_1, w_2 = \lambda_2 e^{i\theta} z_2$ for some $\lambda_1, \lambda_2 > 0$. Then it follows from (1.33) that

$$\begin{aligned} & (1+\tfrac{\lambda_1}{2})Im(z_2\overline{z_1}) + (1+\tfrac{1}{2\lambda_1})\lambda_1\lambda_2 Im(z_2\overline{z_1}) + \tfrac{1}{2}(\lambda_1+\lambda_2)Im(z_2\overline{z_1}) \\ = & (1+\tfrac{\lambda_2}{2})Im(z_2\overline{z_1}) + (1+\tfrac{1}{2\lambda_2})\lambda_1\lambda_2 Im(z_2\overline{z_1}) + \tfrac{1}{2}(\lambda_1+\lambda_2)Im(z_2\overline{z_1}) \\ = & (1+\lambda_1)(1+\lambda_2)Im(z_2\overline{z_1}) \\ = & 0 \end{aligned} \tag{1.37}$$

at the points (z_1, w_1) and (z_2, w_2), which implies $Im(z_2\overline{z_1}) = 0$ and furthermore $Im(w_2\overline{w_1}) = 0$. Thus z_1 and z_2, w_1 and w_2, are colinear, which implies that P equals a plane spanned by one vector from $\{(z_1, 0), (z_2, 0)\}$ and the other from $\{(0, w_1), (0, w_2)\}$. Thus $P \in \mathbb{T}^{\mathbf{2}}$.

The last case is the negative to the first one and the second one. It gives $Im(z_2\overline{z_1}) = Im(w_2\overline{w_1}) = 0$ and $w_2\overline{z_1} - w_1\overline{z_2} = 0$ in (1.25) because of the linear independence, but the former implies the latter by linear transformation, so it is brought down to $Im(z_2\overline{z_1}) = Im(w_2\overline{w_1}) = 0$. Thus we have $P \in \mathbb{T}^{\mathbf{2}}$ by the second case, and that concludes the proof. □

2. Lagrangians of L^p Space

In this section, we'll consider the general case.

For $1 \leq p < \infty$ and $p \neq 2$, we have the following

Theorem 2.1. *The set of Lagrangian subspaces of $\mathbb{C}^2$ with complex L^p norm, $1 \leq p < \infty$ and $p \neq 2$, is where*

$$\mathbb{T}^{\mathbf{2}} := \{span((z,0),(0,w)) : z, w \in U(1)\} \cong U(1) \times U(1) \tag{2.1}$$

and

$$\mathbb{T}^1 := \{P : P = \{\lambda(z,w) : \lambda \in \mathbb{R}, z, w \in U(1),\ zw \textit{ is a constant in } U(1)\}\} \cong U(1). \tag{2.2}$$

Proof. The proof for this claim is a generalization of the one for Theorem 1.6.

As we showed in the first part of the proof to Theorem 1.6,

$$P = \{\lambda(z,w) : \lambda \in \mathbb{R}, z, w \in U(1),\ zw \text{ is a constant in } U(1)\} \tag{2.3}$$

is identical to some

$$P' := span((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1^2\overline{z_2}}{|z_1|^2}e^{i\theta})) \tag{2.4}$$

where $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

Suppose κ_p vanishes on a plane P spanned by (z_1, w_1) and (z_2, w_2). It follows from (1.12) that

$$\begin{aligned}(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{w}{z}|^p)Im(z_2\overline{z_1}) + (1+|\tfrac{z}{w}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{z}{w}|^p)Im(w_2\overline{w_1}) \\ +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Im(\tfrac{z}{w}(w_2\overline{z_1} - w_1\overline{z_2})) &= 0\end{aligned} \tag{2.5}$$

for any $(z, w) \in span((z_1, w_1), (z_2, w_2))$. Analogous to Theorem 1.6, we have three cases to consider.

The first case is that $|\frac{w}{z}| = \lambda$ for some fixed $\lambda > 0$. let $(z, w) = \lambda_1(z_1, w_1) + \lambda_2(z_2, w_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$, then $|\lambda_1 w_1 + \lambda_2 w_2| = \lambda|\lambda_1 z_1 + \lambda_2 z_2|$, that implies $|w_1| = \lambda|z_1|$, $|w_2| = \lambda|z_2|$ and $Re(w_1\overline{w_2}) = \lambda^2 Re(z_1\overline{z_2})$. It follows that either $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$, or $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta}\frac{z_1^2\overline{z_2}}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$.

In the sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta} z_2$ for some $\theta \in [0, 2\pi)$, by (2.5) we have

$$\begin{aligned} & (1+\lambda^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}\lambda^p)Im(z_2\overline{z_1}) + (1+\tfrac{1}{\lambda^p})^{\frac{2}{p}-2}(1+\tfrac{p}{2}\tfrac{1}{\lambda^p})\lambda^2 Im(z_2\overline{z_1}) \\ & \qquad +2(1-\tfrac{p}{2})(1+\lambda^p)^{\frac{2}{p}-2}\lambda^p Im(z_2\overline{z_1}) \\ = & (1+\lambda^p)^{\frac{2}{p}-2}(1+2\lambda^p+\lambda^{2p})Im(z_2\overline{z_1}) \\ = & (1+\lambda^p)^{\frac{2}{p}-2}(1+\lambda^p)^2 Im(z_2\overline{z_1}) \\ = & 0, \end{aligned} \tag{2.6}$$

which gives $Im(z_2\overline{z_1}) = 0$ since $p \geq 1$, and then $Im(w_2\overline{w_1}) = 0$, that means (z_1, w_1) and (z_2, w_2) are colinear. So this case is not allowed.

However, for the other sub-case of $w_1 = \lambda e^{i\theta} z_1$, $w_2 = \lambda e^{i\theta}\frac{z_1^2\overline{z_2}}{|z_1|^2}$ for some $\theta \in [0, 2\pi)$, by (2.5) we have

$$\begin{aligned} & (1+\lambda^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}\lambda^p)Im(z_2\overline{z_1}) - (1+\tfrac{1}{\lambda^p})^{\frac{2}{p}-2}(1+\tfrac{p}{2}\tfrac{1}{\lambda^p})\lambda^2 Im(z_2\overline{z_1}) \\ = & (1+\lambda^p)^{\frac{2}{p}-2}(1-\lambda^{2p})Im(z_2\overline{z_1}) \\ = & 0, \end{aligned} \tag{2.7}$$

Then $\lambda = 1$ or $Im(z_2\overline{z_1}) = 0$, but (z_1, w_1) and (z_2, w_2) can not be colinear. So we have $\lambda = 1$ which gives

$$P = span((z_1, z_1 e^{i\theta}), (z_2, \frac{z_1^2\overline{z_2}}{|z_1|^2}e^{i\theta})), \tag{2.8}$$

where $z_1, z_2 \in \mathbb{C}\setminus\{0\}$ and $Im(z_1\overline{z_2}) \neq 0$ for some $\theta \in [0, 2\pi)$, that finishes the first case.

The second case is $\frac{w}{z}|\frac{w}{z}| = e^{i\theta}$ for some fixed $\theta \in [0, 2\pi)$. Let $w_1 = \lambda_1 e^{i\theta} z_1, w_2 = \lambda_2 e^{i\theta} z_2$ for some $\lambda_1, \lambda_2 > 0$. Then it follows from (2.5) that

$$\begin{aligned}
& (1+\lambda_1^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}\lambda_1^p)Im(z_2\overline{z_1}) + (1+\tfrac{1}{\lambda_1^p})^{\frac{2}{p}-2}(1+\tfrac{p}{2}\tfrac{1}{\lambda_1^p})\lambda_1\lambda_2 Im(z_2\overline{z_1}) \\
& \qquad +(1-\tfrac{p}{2})(1+\lambda_1^p)^{\frac{2}{p}-2}\lambda_1^{p-1}(\lambda_1+\lambda_2)Im(z_2\overline{z_1}) \\
= & (1+\lambda_1^p)^{\frac{2}{p}-2}(1+\lambda_1^p+\lambda_1^{p-1}\lambda_2+\lambda_1^{2p-1}\lambda_2)Im(z_2\overline{z_1}) \\
= & (1+\lambda_1^p)^{\frac{2}{p}-1}(1+\lambda_1^{p-1}\lambda_2)Im(z_2\overline{z_1}) \\
= & 0
\end{aligned} \tag{2.9}$$

and

$$(1+\lambda_2^p)^{\frac{2}{p}-1}(1+\lambda_2^{2p-1}\lambda_1)Im(z_2\overline{z_1}) = 0 \tag{2.10}$$

at the points (z_1, w_1) and (z_2, w_2), which implies $Im(z_2\overline{z_1}) = 0$ and furthermore $Im(w_2\overline{w_1}) = 0$. Thus z_1 and z_2, w_1 and w_2, are colinear, which implies that P equals a plane spanned by one vector from $\{(z_1, 0), (z_2, 0)\}$ and the other from $\{(0, w_1), (0, w_2)\}$. Thus $P \in \mathbb{T}^{\mathbf{2}}$.

The last case is the negative to the first one and the second one. It gives $Im(z_2\overline{z_1}) = Im(w_2\overline{w_1}) = 0$ and $w_2\overline{z_1} - w_1\overline{z_2} = 0$ in (2.5) because of the linear independence, but the former implies the latter by linear transformation, so it is brought down to $Im(z_2\overline{z_1}) = Im(w_2\overline{w_1}) = 0$. Thus we have $P \in \mathbb{T}^{\mathbf{2}}$ by the second case.

Thus the claim follows. □

Remark 2.2. Comparing the results from Theorem 2.1 and Theorem 1.4, we know $(\mathbb{T}^{\mathbf{2}} \cup \mathbb{T}^1) \subset \widetilde{\mathbf{T}}$ since

$$Im(\lambda^2 z_2\overline{z_1}) = Im(\lambda^2 w_1\overline{w_2}) \tag{2.11}$$

if $z_1 w_1 = z_2 w_2$, where $z_1, w_1, z_2, w_2 \in U(1)$, so the set of Lagrangian subspaces of $\mathbb{C}^2$ with complex L^2 norm is the largest among complex L^p norms, $1 \le p < \infty$.

Furthermore, let's see an example, a Lagrangian plane that is for L^2 but not for L^p, $1 \le p < \infty$, $p \ne 2$.

Example 2.3. Suppose $(z_1, w_1) = (2, i)$ and $(z_2, w_2) = (i, 2)$. They satisfy that

$$Im(z_2\overline{z_1}) + Im(w_2\overline{w_1}) = 2i - 2i = 0 \tag{2.12}$$

in (1.21), so *span*$((2, i), (i, 2))$ is a Lagrangian plane for L^2 . However, plugging them in

$$\begin{aligned}
\kappa_p = & (1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{w}{z}|^p)Im(z_2\overline{z_1}) + (1+|\tfrac{z}{w}|^p)^{\frac{2}{p}-2}(1+\tfrac{p}{2}|\tfrac{z}{w}|^p)Im(w_2\overline{w_1}) \\
& \qquad +(1-\tfrac{p}{2})(1+|\tfrac{w}{z}|^p)^{\frac{2}{p}-2}|\tfrac{w}{z}|^p Im(\tfrac{z}{w}(w_2\overline{z_1}-w_1\overline{z_2})),
\end{aligned} \tag{2.13}$$

we have that

$$\begin{aligned}
\kappa_p &= (1+2^p)^{\frac{2}{p}-2}((1+\tfrac{p}{2}\cdot 2^p)\cdot 2 + 2^{2p-2}(1+\tfrac{p}{2}\cdot\tfrac{1}{2^p})\cdot(-2) \\
&\quad +(1-\tfrac{p}{2})\cdot 2^p\cdot\tfrac{3}{4}) \\
&= 2(1+2^p)^{\frac{2}{p}-2}(1-\tfrac{p}{2}\cdot 2^{p-2}+(\tfrac{3}{4}+\tfrac{p}{8})\cdot 2^p - 2^{2p-2}) \qquad (2.14)\\
&= 2(1+2^p)^{\frac{2}{p}-2}(1+3\cdot 2^{p-2}-2^{2p-2}) \\
&= 2(1+2^p)^{\frac{2}{p}-2}(1+3\cdot 2^{p-2}-2^{2p-2})
\end{aligned}$$

at the point $(i, 2)$. Now let

$$f(p) = 1 + 3\cdot 2^{p-2} - 2^{2p-2}, \qquad (2.15)$$

then

$$f'(p) \;=\; \ln 2\cdot 2^{p-2}(3-2^{p+1}) < 0. \qquad (2.16)$$

Hence $f(p)$ is strictly decreasing. But $f(1) = \frac{3}{2}$ and $f(2) = 0$, and so $p = 2$ is the only zero of $f(p)$ for $1 \le p < \infty$. Thus, by (2.14), we conclude that $\kappa_p = 0$ only if $p = 2$.

Acknowledgement. Thanks to Dr. Joseph H. G. Fu's helpful advices and comments to this subject.

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