INTEGRAL GEOMETRY ON MINKOWSKI $p-$SPACE

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ABSTRACT. We give explicit integral geometry formulas on Minkowski $p-$space, $1 < p < \infty$.

1. SPHEREBUNDLE OF MINKOWSKI $p-$SPACE

On Minkowski $p-$Space $\left(\mathbb{R}^2, ||\cdot||_p\right)$, where $||(x,y)||_p = (x^p + y^p)^{1/p}$, the dual norm is $||\cdot||_{\frac{p}{p-1}}$. we know the unit ball B in this space is

$$\left\{(x,y) \in \mathbb{R}^2 : (x^p + y^p)^{1/p} \leq 1\right\},$$

and the support function is

$$h_B : (\mathbb{R}^2)^* \to \mathbb{R}$$
$$h_B(x) = sup\left\{|y(x)| : y(x) \leq 1\right\}.$$

There is a transformation

$$\nabla h_B : S\mathbb{R}^2 \to S^*\mathbb{R}^2$$
$$\nabla h_B((x,y);(\alpha,\beta)) = ((x,y);(\alpha^{p-1},\beta^{p-1})), ||(\alpha,\beta)||_p = 1.$$

And we have the following diagram:

$$\overline{Gr_1(\mathbb{R}^{\mathbf{2}})} \overset{\pi}{\leftarrow} \mathrm{S}\mathbb{R}^2 \overset{\nabla h_B}{\rightarrow} \mathrm{S}^*\mathbb{R}^2$$

The canonical symplectic form ω on the cotangent bundle $S^*\mathbb{R}^2$ on $\mathbb{R}^2$, it can be written explicitly as $\omega_0 = dx \wedge d\xi + dy \wedge d\eta$. The space of geodesics $\overline{Gr_1(\mathbb{R}^{\mathbf{2}})}$ in $\left(\mathbb{R}^2, ||\,.\,||_p\right)$ can represented as $\left\{(x,y),(\alpha,\beta)) : (x,y) \in \mathbb{R}^2, ||(\alpha,\beta)||_p = 1\right\}$, where $((x,y),(\alpha,\beta))$ represents the straight line l passing through (x,y) in the Minkowski space, and $(\alpha,\beta)$is the $p-$norm unit dirction vector of the straight line l. By formal computations and Liouville's Theorem on symplectic structure([Bes]), we have the following lemma:

Lemma 1.1. *$\pi^*((p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dy \wedge d\beta) = (\nabla h_B)^*(\omega_0)$,where π is the projection*

Proof. Since $\nabla h_B(x,(\alpha,\beta)) = ((x,y);(\alpha^{p-1},\beta^{p-1})) = ((x,y);(\xi,\eta))$,

$$dx \wedge d\xi + dy \wedge d\eta = (p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dx \wedge d\beta.$$

Then, by Liouville's Theorem on symplectic structure, we have the equality and $(p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dx \wedge d\beta$ is a symplectic form on $\overline{Gr_1(\mathbb{R}^{\mathbf{2}})}$. □

Now let $\omega = (p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dy \wedge d\beta$, and the 2-density $\phi = |\omega|$ on $\overline{Gr_1(\mathbb{R}^{\mathbf{2}})}$.

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2. Gelfand Tranform of density

Consider the double fibration on $\left(\mathbb{R}^2, ||\cdot||_p\right)$ and $\overline{Gr_1(\mathbb{R}^2)}$:

$$\mathbb{R}^2 \overset{\pi_1}{\leftarrow} \mathcal{I} \overset{\pi_2}{\rightarrow} \overline{Gr_1(\mathbb{R}^2)},$$

where

$$\mathcal{I} = \left\{((x,y), l((x,y);(\alpha,\beta))) : (x,y) \in \mathbb{R}^2, l((x,y);(\alpha,\beta))) \in \overline{Gr_1(\mathbb{R}^2)}\right\}$$

is a set of incidence relations, π_1 and π_2 are natural projections. Since ϕ is a density on $\overline{Gr_1(\mathbb{R}^2)}$, we have the Gelfand transform $GT(\phi)$.

Proposition 2.1. *Considering the Gelfand transform of ϕ, we have*

$$GT(\phi)((x,y);(v_1,v_2)) = 4||(v_1,v_2)||_p,$$

where $\phi = |(p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dy \wedge d\beta|$ and $\alpha^p + \beta^p = 1$, for any vector $(v_1,v_2) \in T_{(x,y)}\mathbb{R}^2$.

Proof. The tangent space of $\mathcal{I}$

$$T_{(x,y,\alpha,\beta)}\mathcal{I} = span\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \alpha}\right\},$$

therefore we have

$$\begin{aligned}
GT(\varphi)((x,y);(v_1,v_2)) &= \int\limits_{\pi_1^{-1}((x,y))} \pi_2^*\phi(((\frac{\partial}{\partial\alpha}v_1 + \frac{\partial}{\partial y}v_2, \frac{\partial}{\partial\alpha})) \\
&= 2\int_{-1}^{1} |\ ((p-1)\alpha^{p-2}dx \wedge d\alpha \\
&\qquad +(p-1)\beta^{p-2}dy \wedge d\beta)(\frac{\partial}{\partial\alpha}v_1 + \frac{\partial}{\partial y}v_2, \frac{\partial}{\partial\alpha})|d\alpha \\
&= 2\int_{-1}^{1} |(p-1)\alpha^{p-2}v_1 \\
&\qquad +(p-1)(1-\alpha^p)^{\frac{p-2}{p}}v_2\frac{d}{d\alpha}(1-\alpha^p)^{\frac{1}{p}}|d\alpha \\
&= 2\int_{-1}^{1} |(p-1)\alpha^{p-2}v_1 - (p-1)(1-\alpha^p)^{-\frac{1}{p}}\alpha^{p-1}v_2|d\alpha.
\end{aligned}$$

The integral above can be split into four parts,

$$\begin{aligned}
I_1 &= 2\int_{-1}^{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}} ((p-1)\alpha^{p-2}v_1 - (p-1)(1-\alpha^p)^{-\frac{1}{p}}\alpha^{p-1}v_2)d\alpha \\
&= [v_1\alpha^{p-1}]_{-1}^{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}} + [v_2((1-\alpha^p)^{\frac{p-1}{p}}]_{-1}^{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}} \\
&= 2(v_1 - \frac{v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}) + \frac{2v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}},
\end{aligned}$$

$$\begin{aligned} I_2 &= 2\int_{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}^{0}((p-1)\alpha^{p-2}v_1-(p-1)(1-\alpha^p)^{-\frac{1}{p}}\alpha^{p-1}v_2)d\alpha \\ &= [v_1\alpha^{p-1}]_{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}^{0}+[v_2((1-\alpha^p)^{\frac{p-1}{p}}]_{-\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}^{0} \\ &= \frac{2v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}+2(v_2-\frac{v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}), \end{aligned}$$

$$\begin{aligned} I_3 &= 2\int_{0}^{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}((p-1)\alpha^{p-2}v_1-(p-1)(1-\alpha^p)^{-\frac{1}{p}}\alpha^{p-1}v_2)d\alpha \\ &= [v_1\alpha^{p-1}]_{0}^{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}+[v_2((1-\alpha^p)^{\frac{p-1}{p}}]_{0}^{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}} \\ &= \frac{2v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}+2(\frac{v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}-v_2), \end{aligned}$$

and

$$\begin{aligned} I_4 &= 2\int_{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}^{1}(-(p-1)\alpha^{p-2}v_1+(p-1)(1-\alpha^p)^{-\frac{1}{p}}\alpha^{p-1}v_2)d\alpha \\ &= 2[v_1\alpha^{p-1}]_{1}^{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}}+2[v_2((1-\alpha^p)^{\frac{p-1}{p}}]_{1}^{\frac{v_1}{(v_1^p+v_2^p)^{\frac{1}{p}}}} \\ &= 2(\frac{v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}-v_1)+\frac{2v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}. \end{aligned}$$

Hence

$$\begin{aligned} GT(\varphi)((x,y);(v_1,v_2)) &= I_1+I_2+I_3+I_4 \\ &= 2(v_1-\frac{v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}})+\frac{2v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}} \\ &\quad +\frac{2v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}+2(v_2-\frac{v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}) \\ &\quad +\frac{2v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}+2(\frac{v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}-v_2) \\ &\quad +2(\frac{v_1^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}}-v_1)+\frac{2v_2^p}{(v_1^p+v_2^p)^{\frac{p-1}{p}}} \\ &= 4(v_1^p+v_2^p)^{\frac{1}{p}}. \end{aligned}$$

Thus we have shown the transformation formula. □

3. Main Formulas

Theorem 3.1. *Given a curve c differentiable almost everywhere in Minkowski $p-$space $\left(\mathbb{R}^2, ||\cdot||_p\right)$, then we have*

$$Length(c) = \frac{1}{4}\int_{l(r,\alpha)\in\overline{Gr_1(\mathbb{R}^2)}} \chi(c\cap l((x,y);\alpha))(p-1)(\alpha^{p-2}dxd\alpha - \frac{\alpha^{p-1}}{\beta}dyd\alpha),$$

where $l((x,y);\alpha))$ is parametrized in the way described in Section 2. Particularly, the Crofton measure for 1-Holmes-Thompson volume is $(p-1)(\alpha^{p-2}dxd\alpha - \frac{\alpha^{p-1}}{\beta}dyd\alpha)$.

Proof. First, we have

$$\begin{aligned}\phi &= (p-1)(\alpha^{p-2}dxd\alpha + \beta^{p-2}dyd\beta)\\ &= (p-1)(\alpha^{p-2}dxd\alpha - \beta^{p-2}\frac{\alpha^{p-1}}{\beta^{p-1}}dyd\alpha)\\ &= (p-1)(\alpha^{p-2}dxd\alpha - \frac{\alpha^{p-1}}{\beta}dyd\alpha),\end{aligned}$$

$$\phi = (p-1)(\alpha^{p-2}dxd\alpha + \beta^{p-2}dyd\beta)$$

By the Theorem of Gelfand transform for Crofton formula (see [AF]), we know

$$\int_{l(r,\alpha)\in\overline{Gr_1(\mathbb{R}^2)}} \chi(c\cap l(r,\alpha))\phi = \int_c GT(\phi).$$

From Proposition 2.1, we have

$$\int_c GT(\phi) = 4Length(c).$$

Thus we have proved the formula. □

There is a natural description of the straight lines in $\overline{Gr_1(\mathbb{R}^2)}$. For a straight line $l(r,\Theta)$ passing through (x,y), let r be the $p-$distance of a straight line l to the origin in the Minkowski space, and (Θ,Ω) be the $p-$norm unit vector that point towards the point at which l is tangent to the sphere $S_r = \left\{(x,y)\in\mathbb{R}^2 : x^p + y^p = r^p\right\}$.

Theorem 3.2. *There exsists a crofton measure*

$$\varphi(\Theta)drd\Theta = \frac{(p-1)^2\Theta^{p(p-2)}|\Omega|^{p^2-3p+1}}{||(\Theta,\Omega)||_{p(p-1)}^{(p-1)(2p-1)}}drd\Theta,$$

where $\Omega = (1-\Theta^p)^{\frac{1}{p}}$, such that

$$Length(c) = \int_{l(r,\Theta)\in\overline{Gr_1(\mathbb{R}^2)}} \chi(c\cap l(r,\Theta))\varphi(\Theta)drd\Theta,$$

for any curve c differentiable almost everywhere in Minkowski $p-$space $\left(\mathbb{R}^2, ||\cdot||_p\right)$.

Proof. By some analytic geometry computations, we have

$$r = \Theta^{p-1}x + \Omega^{p-1}y,$$

and we have the relation

$$(\Theta,\Omega)=(\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}}+\beta^{\frac{p}{p-1}})^{\frac{1}{p}}},-\frac{\alpha^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}}+\beta^{\frac{p}{p-1}})^{\frac{1}{p}}}),$$

therefore

$$\begin{aligned}
dr\wedge d\Theta &= (\Theta^{p-1}dx+\Omega^{p-1}dy)\wedge d\Theta\\
&= \Theta^{p-1}dx\wedge d(\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}}+\beta^{\frac{p}{p-1}})^{\frac{1}{p}}})+\Omega^{p-1}dy\wedge d(\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}}+\beta^{\frac{p}{p-1}})^{\frac{1}{p}}})\\
&= \Theta^{p-1}dx\wedge d(\frac{1}{((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{\frac{1}{p}}})+\Omega^{p-1}dy\wedge d(\frac{1}{((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{\frac{1}{p}}})\\
&= \Theta^{p-1}dx\wedge(-\frac{1}{p})((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\frac{p}{p-1}(\frac{\alpha}{\beta})^{\frac{1}{p-1}}\frac{\beta d\alpha-\alpha d\beta}{\beta^2}\\
&\quad +\Omega^{p-1}dy\wedge(-\frac{1}{p})((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\frac{p}{p-1}(\frac{\alpha}{\beta})^{\frac{1}{p-1}}\frac{\beta d\alpha-\alpha d\beta}{\beta^2}\\
&= (-\frac{1}{p-1})(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}(\Theta^{p-1}dx\wedge(\frac{1}{\beta}d\alpha-\frac{\alpha}{\beta^2}d\beta)\\
&\qquad +\Omega^{p-1}dy\wedge(\frac{1}{\beta}d\alpha-\frac{\alpha}{\beta^2}d\beta)\\
&= (-\frac{1}{p-1})(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\\
&\quad (\Theta^{p-1}(\frac{1}{\beta}+\frac{\alpha^p}{\beta^{p+1}})dx\wedge d\alpha+\Omega^{p-1}(-\frac{1}{\beta}\frac{\beta^{p-1}}{\alpha^{p-1}}-\frac{\alpha}{\beta^2})dy\wedge d\beta)\\
&= (-\frac{1}{p-1})(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\\
&\qquad (\frac{\Theta^{p-1}}{\beta^{p+1}}dx\wedge d\alpha-\frac{\Omega^{p-1}}{\alpha^{p-1}\beta^2}dy\wedge d\beta)\\
&= (-\frac{1}{(p-1)^2})(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\\
&\quad (\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}}(p-1)\alpha^{p-2}dx\wedge d\alpha-\frac{\Omega^{p-1}}{\alpha^{p-1}\beta^{p}}(p-1)\beta^{p-2}dy\wedge d\beta,
\end{aligned}$$

since

$$\frac{\Theta^{p-1}}{\Omega^{p-1}}=-\frac{\beta}{\alpha},$$

then we have

$$\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}}=-\frac{\Omega^{p-1}}{\alpha^{p-1}\beta^{p}}.$$

Hence

$$\begin{aligned}
dr\wedge d\Theta &= -\frac{1}{(p-1)^2}(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} \\
&= ((p-1)\alpha^{p-2}dx\wedge d\alpha+(p-1)\beta^{p-2}dy\wedge d\beta) \\
&= -\frac{1}{(p-1)^2}(\frac{\alpha}{\beta})^{\frac{1}{p-1}}((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{-\frac{p+1}{p}}\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}}\omega \\
&= -\frac{1}{(p-1)^2}\frac{(\frac{\alpha}{\beta})^{\frac{1}{p-1}}\Theta^{p-1}}{((\frac{\alpha}{\beta})^{\frac{p}{p-1}}+1)^{\frac{p+1}{p}}\beta^{p+1}\alpha^{p-2}}\omega \\
&= \frac{1}{(p-1)^2}\frac{\frac{\Omega}{\Theta}\Theta^{p-1}}{((-\frac{\Omega}{\Theta})^p+1)^{\frac{p+1}{p}}(\frac{\Theta^{p-1}}{||(\Theta^{p-1},\Omega^{p-1})||_p})^{p+1}(\frac{\Omega^{p-1}}{||(\Theta^{p-1},\Omega^{p-1})||_p})^{p-2}}\omega \\
&= \frac{1}{(p-1)^2}\frac{\Omega\Theta^p\Theta^{p-1}}{(\frac{\Theta^{p-1}}{||(\Theta^{p-1},\Omega^{p-1})||_p})^{p+1}(\frac{\Omega^{p-1}}{||(\Theta^{p-1},\Omega^{p-1})||_p})^{p-2}}\omega \\
&= \frac{1}{(p-1)^2}\frac{\Omega\Theta^p\Theta^{p-1}||(\Theta^{p-1},\Omega^{p-1})||_p^{2p-1}}{\Theta^{(p-1)(p+1)}\Omega^{(p-1)(p-2)}}\omega \\
&= \frac{1}{(p-1)^2}\frac{||(\Theta^{p-1},\Omega^{p-1})||_p^{2p-1}}{\Theta^{p(p-2)}\Omega^{p^2-3p+1}}\omega \\
&= \frac{||(\Theta,\Omega)||_{p(p-1)}^{(p-1)(2p-1)}}{(p-1)^2\Theta^{p(p-2)}\Omega^{p^2-3p+1}}\omega.
\end{aligned}$$

Now let

$$\varphi(\Theta)=\frac{(p-1)^2\Theta^{p(p-2)}|\Omega|^{p^2-3p+1}}{||(\Theta,\Omega)||_{p(p-1)}^{(p-1)(2p-1)}},$$

where $\Omega=(1-\Theta^p)^{\frac{1}{p}}$, we then have

$$\phi=\varphi(\Theta)drd\Theta.$$

Thus we have shown the formula. □

4. Remark

For $p=1$ or $p=\infty$, the map ∇h_B is not a bijection between the sphere-bundle and cosphere-bundle of $(\mathbb{R}^2,||\cdot||_p)$, it will be interesting to explore those two cases.

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