

ON THE RAUCH COMPARISON THEOREM AND ITS APPLICATIONS

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ABSTRACT. In these notes, we give an exposition on the renowned Rauch comparison theorem in Riemannian geometry and its application.

1. A WARM-UP EXERCISE FROM CALCULUS

Exercise 1.1. Show that $\frac{1}{2} \sin 2t \geq \frac{1}{3} \sin 3t$ on $[0, \frac{\pi}{3}]$.

Proof. Let $f(t) := \frac{1}{2} \sin 2t - \frac{1}{3} \sin 3t$, then $f'(t) = \cos 2t - \cos 3t \geq 0$ on $[0, \frac{\pi}{3}]$. Therefore $f(t)$ is increasing on $[0, \frac{\pi}{3}]$. So $f(t) \geq f(0) = 0$ for $t \in [0, \frac{\pi}{3}]$.

Moreover, we can verify the claim by Figure 1.1 on page 1. □

2. STURM'S THEOREM IN ORDINARY DIFFERENTIAL EQUATION

The following theorem by Sturm was discovered in 1836, and later there were results similar to Sturm's theorem in ordinary differential equation systems and partial differential equations by Picone and Bocher (see [DM]).

Theorem 2.1 (Sturm). *Let $x_1(t)$ and $x_2(t)$ be solutions to equations*

$$x_1''(t) + p_1(t)x_1(t) = 0 \tag{2.1}$$

and

$$x_2''(t) + p_2(t)x_2(t) = 0 \tag{2.2}$$

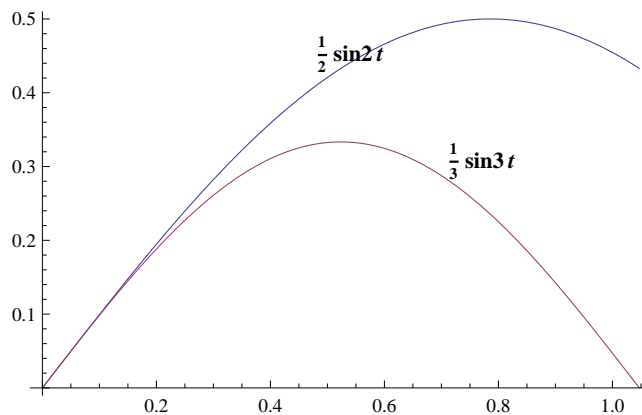


FIGURE 1.1. Graphs of $\frac{1}{2} \sin 2t$ and $\frac{1}{3} \sin 3t$

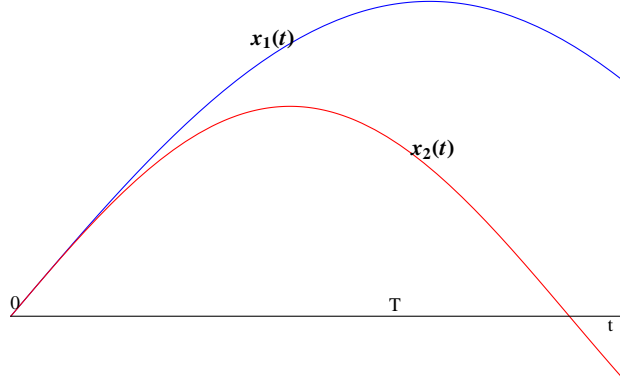


FIGURE 2.1. Comparison of the Solutions to Two ODEs

respectively with initial conditions $x_1(0) = x_2(0) = 0$ and $x_1'(0) = x_2'(0) = 1$, where $p_1(t)$ and $p_2(t)$ are continuous on $[0, T]$. Suppose $p_1(t) \leq p_2(t)$ on $[0, T]$ and $x_2(t) > 0$ on $(0, T]$. Then $x_1(t) \geq x_2(t)$ on $[0, T]$.

Basically, the proof for Theorem 2.1 uses the product rule (or integration by part), l'Hôpital's rule, and some other tools and the geometric intuition can be illustrated by Figure 2.1 on page 2.

Proof. Multiplying (2.1) by $x_2(t)$ and (2.2) by $x_1(t)$, and subtracting one from the other, one has

$$x_1(t)x_2''(t) - x_2(t)x_1''(t) + (p_2(t) - p_1(t))x_1(t)x_2(t) = 0. \quad (2.3)$$

Assume that the first zeroes of $x_1(t)$ is t_1 , $t_1 < T$. Then $x_1'(t_1) < 0$ and $x_2(t_1) > 0$. Integrating (2.3) by part over $[0, t_1]$,

$$\begin{aligned} 0 &= \int_0^{t_1} (x_1(t)x_2''(t) - x_2(t)x_1''(t) + (p_2(t) - p_1(t))x_1(t)x_2(t))dt \\ &= (x_1(t)x_2'(t) - x_2(t)x_1'(t))|_0^{t_1} + \int_0^{t_1} (p_2(t) - p_1(t))x_1(t)x_2(t)dt \\ &= -x_2(t_1)x_1'(t_1) + \int_0^{t_1} (p_2(t) - p_1(t))x_1(t)x_2(t)dt. \end{aligned} \quad (2.4)$$

But $x_1'(t_1) < 0$, $x_2(t_1) > 0$, $p_1(t) \leq p_2(t)$ and $x_1(t), x_2(t) > 0$ on $(0, t_1)$ imply

$$-x_2(t_1)x_1'(t_1) + \int_0^{t_1} (p_2(t) - p_1(t))x_1(t)x_2(t)dt > 0, \quad (2.5)$$

which is a contradiction. So $x_1(t) > 0$ on $(0, T]$. From (2.4) we can see that $x_1(t)x_2'(t) - x_2(t)x_1'(t) \leq 0$, that gives $\frac{x_1'(t)}{x_1(t)} \geq \frac{x_2'(t)}{x_2(t)}$. Therefore, for any small $\varepsilon > 0$ and any $x \in [\varepsilon, T]$,

$$\ln(x_1(t)) - \ln(x_1(\varepsilon)) = \int_\varepsilon^t \frac{x_1'(t)}{x_1(t)} dt \geq \int_\varepsilon^t \frac{x_2'(t)}{x_2(t)} dt = \ln(x_2(t)) - \ln(x_2(\varepsilon)), \quad (2.6)$$

which implies

$$\frac{x_1(t)}{x_2(t)} \geq \frac{x_1(\varepsilon)}{x_2(\varepsilon)} \quad (2.7)$$

for any small $\varepsilon > 0$. Then the claim follows by applying the l'Hôpital's rule. \square

3. RAUCH COMPARISON THEOREM AND ITS APPLICATION

3.1. Some Ingredients.

Definition 3.1 (Jacobi Field). A vector field J along a geodesic γ in a Riemannian manifold is said to be a Jacobi field if it satisfies the Jacobi equation

$$\frac{D^2}{dt^2}J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0, \quad (3.1)$$

where R is the Riemann curvature tensor.

By imposing a parallel transported orthonormal frame $\{e_i(t) : i = 1, \dots, n\}$ along a geodesic γ on a n -dimensional manifold M , in which $e_i(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}$, can transform the Jacobi equation (3.1) to a linear differential equation system.

Example 3.2 (Jacobi Fields on Positive Constant Curvature Manifold). For n -dimensional Riemannian manifolds of constant curvature $K > 0$, the space of Jacobi fields along a geodesic γ is

$$\text{span} \left\{ e_1(t), te_1(t), \left\{ \sin(\sqrt{K}t)e_i(t) \right\}_{i=2}^n, \left\{ \cos(\sqrt{K}t)e_i(t) \right\}_{i=2}^n \right\}. \quad (3.2)$$

Definition 3.3 (Conjugate Point). A point q is said to be a conjugate point to another point p along a geodesic γ in a Riemannian manifold if there exists a non-zero Jacobi field along γ that vanishes at p and q .

Example 3.4. Any point on the sphere S^n is a conjugate point to its antipodes.

3.2. The Rauch Comparison Theorem. Now let's state the Rauch comparison theorem.

Theorem 3.5 (Rauch 1951, [R]). *Let M_1 and M_2 be Riemannian manifolds, $\gamma_1 : [0, T] \rightarrow M_1$ and $\gamma_2 : [0, T] \rightarrow M_2$ be normalized geodesic segments such that $\gamma_2(0)$ has no conjugate points along γ_2 , and J_1, J_2 be Jacobi fields along γ_1 and γ_2 such that $J_1(0) = J_2(0) = 0$ and $|J_1'(0)| = |J_2'(0)|$. Suppose that the sectional curvatures of M_1 and M_2 satisfy $K_1 \leq K_2$ for all 2-planes containing γ_1' and γ_2' on each manifold. Then $|J_1(t)| \geq |J_2(t)|$ for all $t \in [0, T]$.*

Remark 3.6. The condition “normal Jacobi fields along γ_1 and γ_2 ” can be replaced by the Jacobi fields satisfying $\langle J_1'(0), \gamma_1'(0) \rangle = \langle J_2'(0), \gamma_2'(0) \rangle$, that we can see in the proof.

3.3. A Tool for the Proof. In order to prove Rauch comparison theorem, we need to introduce definition of index form of a vector field and a so-called index lemma which will be shown later.

Definition 3.7 (Index Form). The index form of a piecewise differentiable vector field V along a geodesic γ on a Riemannian manifold M is defined as

$$I_t(V, V) := \int_0^t (\langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle) dt. \quad (3.3)$$

It turns out that the index form of a Jacobi field is no larger than an arbitrary piecewise differentiable vector field along a geodesic, precisely,

Lemma 3.8 (Index Lemma). *Let J be a Jacobi field along a geodesic $\gamma : [0, T] \rightarrow M$, which has no conjugate point to $\gamma(0)$ in the interval $(0, T]$, with $\langle J, \gamma' \rangle = 0$, and V be a piecewise differentiable vector field along γ , with $\langle V, \gamma' \rangle = 0$. Suppose that $J(0) = V(0) = 0$ and $J(t_0) = V(t_0)$, $t_0 \in (0, T]$. Then $I_{t_0}(J, J) \leq I_{t_0}(V, V)$.*

3.4. The Proof for the Rauch Comparison Theorem.

Proof of Rauch Comparison Theorem, 3.5. First we claim that

$$\langle J(t), \gamma'(t) \rangle = \langle J'(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle, \quad (3.4)$$

for $i = 1, 2$. Because from the Jacobi equation,

$$\langle J', \gamma' \rangle' = \langle J'', \gamma' \rangle = -\langle R(J, \gamma')\gamma', \gamma' \rangle = 0, \quad (3.5)$$

that implies

$$\langle J(t), \gamma'(t) \rangle' = \langle J'(t), \gamma'(t) \rangle \equiv \langle J'(0), \gamma'(0) \rangle, \quad (3.6)$$

which yields (3.4).

It follows from (3.4) and the assumption $J_1(0) = J_2(0) = 0$, that $\langle J_1'(0), \gamma_1'(0) \rangle = \langle J_2'(0), \gamma_2'(0) \rangle$ is equivalent to $\langle J_1(t), \gamma_1'(t) \rangle = \langle J_2(t), \gamma_2'(t) \rangle$ which means the tangential components of J_1 and J_2 have the same length. So the assumption that $J_i \perp \gamma_i$ for $i = 1, 2$ can be replaced by $\langle J_1'(0), \gamma_1'(0) \rangle = \langle J_2'(0), \gamma_2'(0) \rangle$.

If $|J_1'(0)| = |J_2'(0)| = 0$, then $J_1 = J_2 = 0$. Contrarily, if $|J_1'(0)| = |J_2'(0)| > 0$, then let $f_j(t) := |J_j(t)|^2$ for $i = 1, 2$. Since J_2 does not have any conjugate points on $(0, T]$, $\frac{f_1(t)}{f_2(t)}$ is well-defined for $t \in (0, T]$. Using l'Hôpital's rule twice,

$$\lim_{t \rightarrow 0^+} \frac{f_1(t)}{f_2(t)} = \lim_{t \rightarrow 0^+} \frac{\langle J_1(t), J_1(t) \rangle}{\langle J_2(t), J_2(t) \rangle} = \lim_{t \rightarrow 0^+} \frac{2\langle J_1'(t), J_1(t) \rangle}{2\langle J_2'(t), J_2(t) \rangle} = \lim_{t \rightarrow 0^+} \frac{\langle J_1'(t), J_1'(t) \rangle}{\langle J_2'(t), J_2'(t) \rangle} = 1. \quad (3.7)$$

So it suffices to show that

$$\frac{d}{dt} \left(\frac{f_1(t)}{f_2(t)} \right) \geq 0, \text{ i.e. } f_1' f_2 \geq f_1 f_2'. \quad (3.8)$$

We can choose $t_0 \in (0, T]$ such that $f_1(t_0) > 0$, because otherwise (3.8) would be satisfied trivially. Then let

$$U_i(t) := \frac{1}{\sqrt{f_i(t_0)}} J_i(t) \quad (3.9)$$

for $i = 1, 2$. Therefore, by the Jacobi equations and the assumptions,

$$\begin{aligned} \frac{f_i'(t_0)}{f_i(t_0)} &= \frac{2\langle J_i'(t_0), J_i(t_0) \rangle}{f_i(t_0)} \\ &= 2\langle U_i'(t_0), U_i(t_0) \rangle \\ &= \langle U_i(t_0), U_i(t_0) \rangle' \\ &= \int_0^{t_0} \langle U_i(t), U_i(t) \rangle'' dt \\ &= 2 \int_0^{t_0} (\langle U_i', U_i' \rangle + \langle U_i'', U_i \rangle) dt \\ &= 2 \int_0^{t_0} (\langle U_i', U_i' \rangle - \langle R(\gamma', U_i)\gamma', U_i \rangle) dt \\ &= 2I_{t_0}(U_i, U_i). \end{aligned} \quad (3.10)$$

for $i = 1, 2$. Thus, (3.8) is equivalent to

$$I_{t_0}(U_1, U_1) \geq I_{t_0}(U_2, U_2). \quad (3.11)$$

In order to show (3.11), we set up a parallel orthonormal basis

$$\{e_{i,1}(t), \dots, e_{i,n}(t)\} \quad (3.12)$$

along γ_i , such that

$$e_{i,1}(t) = \frac{\gamma'_i(t)}{|\gamma'_i(t)|} \text{ and } e_{i,2}(t_0) = U_i(t_0) \quad (3.13)$$

for $i = 1, 2$. Then define a map φ from the set of vector fields along γ_1 on M_1 to the ones along γ_2 on M_2 , by

$$\varphi\left(\sum_{j=1}^n g_j(t)e_{1,j}(t)\right) := \sum_{j=1}^n g_j(t)e_{2,j}(t), \quad (3.14)$$

which has the properties that $|(\varphi(U_1))'| = |U_1'|$ and $\langle \varphi(U_1), \varphi(U_1) \rangle = \langle U_1, U_1 \rangle$. Therefore, by the assumptions, that $K_1 \leq K_2$ for all 2-planes containing γ_1' and γ_2' , and γ_1 and γ_2 are normalized, we have

$$I_{t_0}(U_1, U_1) \geq I_{t_0}(\varphi(U_1), \varphi(U_1)), \quad (3.15)$$

and then the claim (3.11) follows from the index lemma, Lemma 3.8. \square

One of the beautiful applications of the Rauch comparison theorem is the sphere theorem by Berger and Klingenberg in 1960s.

3.5. Application of the Rauch Comparison theorem.

Theorem 3.9 (Sphere Theorem, 1960). *Any compact, simply-connected, and strictly $\frac{1}{4}$ -pinched manifold M^n , that is the sectional curvature K of M^n satisfying $0 < \frac{1}{4}K_{max} < K \leq K_{max}$, is homeomorphic to S^n .*

Actually, it is not only “homeomorphic” but also diffeomorphic, that is known as the differentiable sphere theorem proved recently by Brendle and Schoen using another technique.

Theorem 3.10 (Differentiable Sphere Theorem, [BS]). *Any compact, simply connected and strictly $\frac{1}{4}$ -pinched manifold is diffeomorphic to S^n .*

REFERENCES

- [BS] S. Brendle and R. Schoen, Manifolds with $1/4$ -pinched curvature are space forms, J. Amer. Math. Soc. 22, 287–307 (2009).
- [CE] Jeff Cheeger, D. G. Ebin, Comparison Theorems in Riemannian Geometry, Published by AMS Bookstore, 2008.
- [D] Manfredo Perdigão do Carmo, Riemannian geometry, Translated by Francis Flaherty, Edition 4, Published by Birkhäuser, 1992.
- [DM] J. B. Diaz; Joyce R. McLaughlin, Sturm comparison theorems for ordinary and partial differential equations, Bull. Amer. Math. Soc. 75 (1969), 335-339.
- [R] H.E. Rauch, A contribution to differential geometry in the large, Ann. Math. 54 (1951), 38-55.