On the Rauch Comparison Theorem and Its Applications

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A Warm-up Exercise from Calculus

Exercise

Show that $\frac{1}{2}\sin 2t \geq \frac{1}{3}\sin 3t$ on $[0, \frac{\pi}{3}]$.

A Warm-up Exercise from Calculus

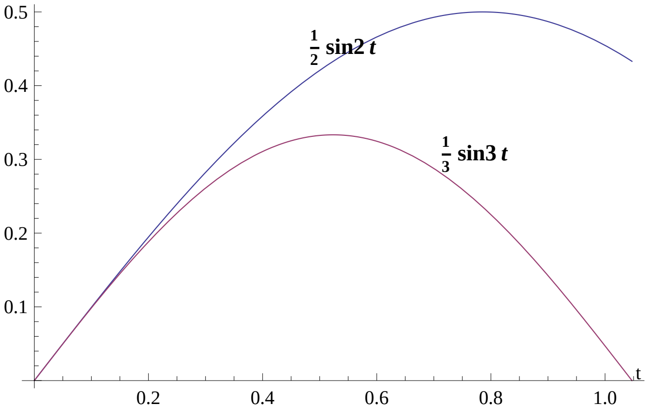

Figure: Graphs of $\frac{1}{2}\sin 2t$ and $\frac{1}{3}\sin 3t$

Sturm's Theorem

Theorem (Sturm)

Let $x_1(t)$ *and* $x_2(t)$ *be solutions to equations*

$$x_1''(t) + p_1(t)x_1(t) = 0 \tag{1}$$

and

$$x_2''(t) + p_2(t)x_2(t) = 0 \tag{2}$$

respectively with initial conditions $x_1(0) = x_2(0) = 0$ *and* $x_1'(0) = x_2'(0) = 1$*, where* $p_1(t)$ *and* $p_2(t)$ are continuous *on* $[0, T]$*. Suppose* $p_1(t) \leq p_2(t)$ *on* $[0, T]$ *and* $x_2(t) > 0$ *on* $(0, T]$*. Then* $x_1(t) \geq x_2(t)$ *on* $[0, T]$*.*

Sturm's Theorem

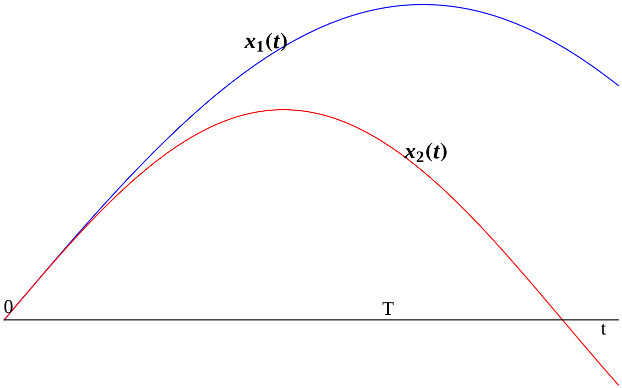

Figure: Comparison of the Solutions to Two ODEs

Jacobi Field

Definition (Jacobi Field)

A vector field J along a geodesic γ in a Riemmannian manifold is said to be a Jacobi field if it satisfies the Jacobi equation

$$\frac{D^2}{dt^2}J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0, \tag{3}$$

where R is the Riemann curvature tensor.

Conjugate Point

Definition (Conjugate Point)

A point q is said to be a conjugate point to another point p along a geodesic γ in a Riemannian manifold if there exists a non-zero Jacobi field along γ that vanishes at p and q.

The Rauch Comparison Theorem

Theorem (Rauch, 1951)

Let M_1 and M_2 be Riemannian manifolds, $\gamma_1 : [0, T] \to M_1$ and $\gamma_2 : [0, T] \to M_2$ be normalized geodesic segments such that $\gamma_2(0)$ has no conjugate points along γ_2, and J_1, J_2 be normal Jacobi fields along γ_1 and γ_2 such that $J_1(0) = J_2(0) = 0$ and $|J_1'(0)| = |J_2'(0)|$. Suppose that the sectional curvatures of M_1 and M_2 satisfy $K_1 \le K_2$ for all 2-planes containing γ_1' and γ_2' on each manifold. Then $|J_1(t)| \ge |J_2(t)|$ for all $t \in [0, T]$.

Remark

The condition "normal Jacobi fields along γ_1 and γ_2 " can be replaced by the Jacobi fields satisfying $\langle J_1'(0), \gamma_1'(0)\rangle = \langle J_2'(0), \gamma_2'(0)\rangle$, that we can see in the proof.

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Index Form

Definition (Index Form)

The index form of a piecewise differentiable vector field V along a geodesic γ on a Riemannian manifold M is defined as

$$I_t(V, V) := \int_0^t (\langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle) dt. \tag{4}$$

Index Lemma

Lemma (Index Lemma)

Let J be a Jacobi field along a geodesic $\gamma : [0, T] \to M$, which has no conjugate point to $\gamma(0)$ in the interval $(0, T]$, with $\langle J, \gamma' \rangle = 0$, and V be a piecewise differentiable vector field along γ, with $\langle V, \gamma' \rangle = 0$. Suppose that $J(0) = V(0) = 0$ and $J(t_0) = V(t_0)$, $t_0 \in (0, T]$. Then $I_{t_0}(J, J) \leq I_{t_0}(V, V)$.

Sphere Theorem

Theorem (Berger-Klingenberg, 1960)

Any compact, simply connected, and strictly $\frac{1}{4}$-pinched manifold M^n, that is the sectional curvature K of M^n satisfying $0 < \frac{1}{4}K_{max} < K \leq K_{max}$, is homeomorphic to S^n.

Differentiable Sphere Theorem

Theorem (Brendle and Schoen, 2009)

Any compact, simply connected and strictly $\frac{1}{4}$-strictly pinched manifold M^n is diffeomorphic to S^n.