# SYMPLECTIC STRUCTURE ON THE SPACE OF AFFINE LINES

LOUIS YANG LIU

We are going to give a detailed introduction on the symplectic structure on the space of affine lines on a plane induced from the natural symplectic structure on the cotangent bundle of plane.

## 1. Compatibility of Pushforwards

Let us consider the natural projection

$$\begin{aligned} \pi : S(\mathbb{R}^2) \to \overline{Gr_1(\mathbb{R}^2)} = \{(r,\theta)\} \\ \pi((x,y,\theta)) = (-x\sin\theta + y\cos\theta, \theta). \end{aligned} \tag{1.1}$$

**Proposition 1.1.** *Let*

$$\begin{aligned} F_t : S\mathbb{R}^2 \to S\mathbb{R}^2 \\ F_t((x,y,\theta)) = (x + t\cos\theta, y + t\sin\theta, \theta), \end{aligned} \tag{1.2}$$

*then the pushforward map from $T_{(x,y,\theta)}S\mathbb{R}^2$ to $T_{(x+t\cos\theta, y+t\sin\theta,\theta)}S\mathbb{R}^2$ induced by $R$-action $F_t(\theta)$ is compatible with the natural projection $\pi$, in other words,*

$$(\pi_*)_{(x,y,\theta)}(v) = (\pi_*)_{(\bar{x},\bar{y},\theta)}(F_{t*})_{(x,y,\theta)}(v),$$

*where $\bar{x} = x + tcos\theta$ and $\bar{y} = y + t\sin\theta$, for any $v \in T_{(x,y,\theta)}S\mathbb{R}^2$.*

*Proof.* Since $\pi = \pi \circ F_t$, then $\pi_* = \pi_* \circ F_{t*}$. So the conclusion follows. □

## 2. Natural Differential Form on Sphere Bundles

The natural symplectic form $d\alpha = dx \wedge d\xi + dy \wedge d\eta$ on $T\mathbb{R}^2$ is induced from the natural symplectic structure on $T^*\mathbb{R}^2$ and the Euclidean metric. Then the inclusion $i : S\mathbb{R}^2 \to T\mathbb{R}^2$, $i(x,y,\theta) = (x, y, \xi = \cos\theta, \eta = \sin\theta)$ induces a two form on $S\mathbb{R}^2$, which is

$$\begin{aligned} \bar{\omega} := i^* d\alpha &= dx \wedge d\xi + dy \wedge d\eta \\ &= dx \wedge d(\cos\theta) + dy \wedge d(\sin\theta) \\ &= -\sin\theta dx \wedge d\theta + \cos\theta dy \wedge d\theta. \end{aligned}$$

We have the following proposition.

**Proposition 2.1.** *For any $v, w \in T_{(x,y,\theta)}S\mathbb{R}^2$,*

$$\bar{\omega}_{(x,y,\theta)}(v,w) = \bar{\omega}_{F_t(x,y,\theta)}((F_{t*})_{(x,y,\theta)}(v), (F_{t*})_{(x,y,\theta)}(w)),$$

*where $\bar{x} = x + tcos\theta$ and $\bar{y} = y + t\sin\theta$ for any $t \in \mathbb{R}$.*

---

*Date*: Mar 20, 2008.

*Proof.* Let $v = k_1\frac{\partial}{\partial x} + k_2\frac{\partial}{\partial y} + l\frac{\partial}{\partial \theta}$ and $w = \bar{k}_1\frac{\partial}{\partial x} + \bar{k}_2\frac{\partial}{\partial y} + \bar{l}\frac{\partial}{\partial \theta}$, then

$$(F_{t*})_{(x,y,\theta)}(v) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} k_1 - tl\sin\theta \\ k_2 + tl\cos\theta \\ l \end{pmatrix}$$

and

$$(F_{t*})_{(x,y,\theta)}(w) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} \bar{k}_1 - t\bar{l}\sin\theta \\ \bar{k}_2 + t\bar{l}\cos\theta \\ \bar{l} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \bar{\omega}_{(\bar{x},\bar{y},\theta)}((F_{t*})_{(x,y,\theta)}(v), (F_{t*})_{(x,y,\theta)}(w)) &= -\sin\theta dx \wedge d\theta((\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} k_1 - tl\sin\theta \\ k_2 + tl\cos\theta \\ l \end{pmatrix}, \\ &\qquad (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} \bar{k}_1 - t\bar{l}\sin\theta \\ \bar{k}_2 + t\bar{l}\cos\theta \\ \bar{l} \end{pmatrix}) \\ &\quad + \cos\theta dy \wedge d\theta((\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} k_1 - tl\sin\theta \\ k_2 + tl\cos\theta \\ l \end{pmatrix}, \\ &\qquad (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta}) \begin{pmatrix} \bar{k}_1 - t\bar{l}\sin\theta \\ \bar{k}_2 + t\bar{l}\cos\theta \\ \bar{l} \end{pmatrix}) \\ &= -\sin\theta((k_1 - tl\sin\theta)\bar{l} - (\bar{k}_1 - t\bar{l}\sin\theta)l) \\ &\quad + \cos\theta((k_2 + tl\cos\theta)\bar{l} - (\bar{k}_2 - t\bar{l}\cos\theta)l) \\ &= -\sin\theta(k_1\bar{l} - \bar{k}_1 l) + \cos\theta(k_2\bar{l} - \bar{k}_2 l), \end{aligned}$$

which is independent of $t \in \mathbb{R}$.

So we get

$$\bar{\omega}_{(x,y,\theta)}(v, w) = \bar{\omega}_{(\bar{x},\bar{y},\theta)}((F_{t*})_{(x,y,\theta)}(v), (F_{t*})_{(x,y,\theta)}(w)),$$

where $\bar{x} = x + t\cos\theta$ and $\bar{y} = y + t\sin\theta$, for any $t \in \mathbb{R}$. □

For the pushforward induced by the natural projection $\pi : S(\mathbb{R}^2) \to \overline{Gr_1(\mathbb{R}^2)}$, we have the following lemma.

**Lemma 2.2.** $ker((\pi_*)_{(x,y,\theta)}) = \left\{\lambda\cos\theta\frac{\partial}{\partial x} + \lambda\sin\theta\frac{\partial}{\partial y} : \lambda \in \mathbb{R}\right\}$.

*Proof.* For any $v = k_1\frac{\partial}{\partial x} + k_2\frac{\partial}{\partial y} + l\frac{\partial}{\partial \theta} \in T_{(x,y,\theta)}S\mathbb{R}^2$, since

$$(\pi_*)_{(x,y,\theta)}(v) = (\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}) \begin{pmatrix} -k_1\sin\theta + k_2\cos\theta + l(-x\cos\theta - y\sin\theta) \\ l \end{pmatrix}$$

as we did in Proposition 1.1, we know

$$(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}) \begin{pmatrix} -k_1\sin\theta + k_2\cos\theta + l(-x\cos\theta - y\sin\theta) \\ l \end{pmatrix} = 0$$

iff $k_1 = \lambda\cos\theta$, $k_2 = \lambda\sin\theta$ and $l = 0$ for $\lambda \in \mathbb{R}$.

Thus,

$$ker((\pi_*)_{(x,y,\theta)}) = \left\{ \lambda \cos\theta \frac{\partial}{\partial x} + \lambda \sin\theta \frac{\partial}{\partial y} : \lambda \in \mathbb{R} \right\}. \tag{2.1}$$

□

*Remark* 2.3. By this lemma, we have the conclusion that if $(\pi_*)_{(x,y,\theta)}(v) = (\pi_*)_{(x,y,\theta)}(\bar{v})$, then $\bar{v} = v + \lambda_0 \cos\theta \frac{\partial}{\partial x} + \lambda_0 \sin\theta \frac{\partial}{\partial y}$ for some $\lambda_0 \in \mathbb{R}$.

We have another lemma.

**Lemma 2.4.** *For any* $v \in ker((\pi_*)_{(x,y,\theta)})$, $\bar{\omega}_{(x,y,\theta)}(v, \cdot) = 0$.

By Lemma 2.2, let $v = \lambda \cos\theta \frac{\partial}{\partial x} + \lambda \sin\theta \frac{\partial}{\partial y}$, then for any $w = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + l \frac{\partial}{\partial \theta} \in T_{(x,y,\theta)} S\mathbb{R}^2$, we have

$$\begin{aligned} \bar{\omega}_{(x,y,\theta)}(v, w) &= -\sin\theta(-\lambda l \cos\theta) + \cos\theta(-\lambda l \sin\theta) \\ &= 0. \end{aligned}$$

Furthermore we have the following proposition.

**Proposition 2.5.** *Suppose*

$$(\pi_*)_{(x,y,\theta)}(v) = (\pi_*)_{(x,y,\theta)}(\bar{v})$$

*and*

$$(\pi_*)_{(x,y,\theta)}(w) = (\pi_*)_{(x,y,\theta)}(\bar{w}),$$

where $v, w, \bar{v}, \bar{w} \in T_{(x,y,\theta)} S\mathbb{R}^2$, then

$$\bar{\omega}_{(x,y,\theta)}(v, w) = \bar{\omega}_{(x,y,\theta)}(\bar{v}, \bar{w}).$$

*Proof.* Let $v = k_1 \frac{\partial}{\partial x} + k_2 \frac{\partial}{\partial y} + l \frac{\partial}{\partial \theta}$ and $w = \bar{k}_1 \frac{\partial}{\partial x} + \bar{k}_2 \frac{\partial}{\partial y} + \bar{l} \frac{\partial}{\partial \theta}$, by Remark 2.3, we know

$$\bar{v} = v + \lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}$$

and

$$\bar{w} = w + \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Therefore

$$\begin{aligned} \bar{\omega}_{(x,y,\theta)}(\bar{v}, \bar{w}) &= \bar{\omega}_{(x,y,\theta)}(v + \lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}, w + \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}) \\ &= \bar{\omega}_{(x,y,\theta)}(v, w) + \bar{\omega}_{(x,y,\theta)}(v, \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}) \\ &\quad + \bar{\omega}_{(x,y,\theta)}(\lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}, w) \\ &\quad + \bar{\omega}_{(x,y,\theta)}(\lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}, \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}) \\ &= \bar{\omega}_{(x,y,\theta)}(v, w) + \bar{\omega}_{(x,y,\theta)}(v, \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}) \\ &\quad + \bar{\omega}_{(x,y,\theta)}(\lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}, w). \end{aligned}$$

By Lemma 2.4,

$$\bar{\omega}_{(x,y,\theta)}(v, \lambda_2 \cos\theta \frac{\partial}{\partial x} + \lambda_2 \sin\theta \frac{\partial}{\partial y}) + \bar{\omega}_{(x,y,\theta)}(\lambda_1 \cos\theta \frac{\partial}{\partial x} + \lambda_1 \sin\theta \frac{\partial}{\partial y}, w) = 0. \tag{2.2}$$

Thus we have shown

$$\bar{\omega}_{(x,y,\theta)}(v, w) = \bar{\omega}_{(x,y,\theta)}(\bar{v}, \bar{w}).$$

□

## 3. Symplectic Structure on $\overline{Gr_1(\mathbb{R}^2)}$

Now we can construct a local symplectic structure on $\overline{Gr_1(\mathbb{R}^2)}$ of geodesics in $\mathbb{R}^2$. By Proposition 2.2 and Proposition 2.5, we have a well-defined two form $\omega$ on $\overline{Gr_1(\mathbb{R}^2)}$, $\omega_{(r,\theta)}(\tilde{v}, \tilde{w}) = \bar{\omega}_{(x,y,\theta)}(v, w)$, where $(\pi_*)_{(x,y,\theta)}(v) = \tilde{v}$, $(\pi_*)_{(x,y,\theta)}(w) = \tilde{w}$, and $r = -x\sin\theta + y\cos\theta$. In other words, $\bar{\omega}_{(x,y,\theta)}(v, w)$ is independent of the choices of preimages under the pushforward induced by the projection. Since $\bar{\omega}_{(x,y,\theta)} = -\sin\theta dx \wedge d\theta + \cos\theta dy \wedge d\theta$, $x = r\cos\theta$ and $y = r\sin\theta$, hence

$$\begin{aligned} \omega_{(r,\theta)} &= -\sin\theta\, d(r\cos\theta) \wedge d\theta + \cos\theta\, d(r\sin\theta) \wedge d\theta \\ &= \sin^2\theta\, dr \wedge d\theta + \cos^2\theta\, dr \wedge d\theta \\ &= dr \wedge d\theta. \end{aligned}$$

**Proposition 3.1.** *$\omega_{(r,\theta)} = dr \wedge d\theta$ is a symplectic form on $\overline{Gr_1(\mathbb{R}^2)}$.*

*Proof.* It is clear to us $\omega_{(r,\theta)} = dr \wedge d\theta$ is a skew-symmetric non-degenerate closed form on the 2 dimensional manifold $\overline{Gr_1(\mathbb{R}^2)}$. □

Thus we have finished the construction of a local symplectic structure on the space of geodesics in $\mathbb{R}^2$.

*Acknowledgement.* Thanks to Dr. Joseph H. G. Fu's guidance.

## References

[B] A. Besse, Manifolds All of Whose Geodesics are Closed, Springer-Verlag, Berlin, Heidelberg, New York, 1978. MR 80c:53044