Crofton Measures for Holmes-Thompson Volumes in Minkowski Space

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Minkowski Space and Geodesics

Definition

A function $F : \mathbb{R}^n \to \mathbb{R}$ is a Minkowski norm if
1. $F(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$.
2. $F(\lambda x) = |\lambda| F(x)$ for any $x \in \mathbb{R}^n \setminus \{0\}$.
3. $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the Hessian $H(\frac{1}{2}F^2) > 0$ on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

We denote a Minkowski space by $(\mathbb{R}^n, F)$.

Minkowski Space and Geodesics

From the definition of Minkowski norm, we can infer the following theorem about geodesics in Minkowski space:

Theorem

The straight line joining two points in Minkowski space is the only shortest curve joining them.

Proof.

(Outline) Apply Euler-Lagrange Equation,

$$H(F)\frac{d^2r(t)}{dt^2} = 0.$$

$$\begin{aligned} \frac{1}{2}H(F^2)\frac{d^2r(t)}{dt^2} &= F(r'(t))H(F)\frac{d^2r(t)}{dt^2} + (\nabla F(r'(t))^T\nabla F(r'(t))\frac{d^2r(t)}{dt^2} \\ &= H(F)\frac{d^2r(t)}{dt^2}. \end{aligned}$$

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Symplectic Structures on Cotangent Bundle

Definition

The canonical 1-form α on $T^*\mathbb{R}^n$ is $\alpha_\xi(X) := \xi(\pi_{0*}X)$ for $X \in T_\xi T^*\mathbb{R}^n$, where $\pi_0 : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the natural projection. Then the canonical symplectic form on $T^*\mathbb{R}^n$ is $\omega := d\alpha$.

Fact

dF is a diffeomorphism from $S_x\mathbb{R}^n$ to $S_x^*\mathbb{R}^n$, which induces another diffeomorphism

$$\varphi_F : S\mathbb{R}^n \to S^*\mathbb{R}^n$$
$$\varphi_F((x, \bar{\xi}_x)) = (x, dF(\bar{\xi}_x)).$$

Lemma

The diffeomorphism $\varphi_F : S\mathbb{R}^n \to S^*\mathbb{R}^n$ induces a 2-form $\bar{\omega} = \varphi_F^*(\omega|_{S^*\mathbb{R}^n}) = Hess(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n}$, where $\star$ is the Frobenius inner product matrices, on $S\mathbb{R}^n$.

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Gelfand Transform

Definition

Let $M \overset{\pi_1}{\leftarrow} \mathcal{F} \overset{\pi_2}{\rightarrow} \Gamma$ be double fibration where M and Γ are two manifolds, $\pi_1 : \mathcal{F} \longrightarrow M$ and $\pi_2 : \mathcal{F} \longrightarrow \Gamma$ are two fibre bundles, and $\pi_1 \times \pi_2 : \mathcal{F} \longrightarrow M \times \Gamma$ is an submersion. Let Φ be a density on Γ, then the Gelfand transform of Φ is defined as $GT(\Phi) := \pi_{1*}\pi_2^*\Phi$.

Theorem

Suppose $M_\gamma := \pi_1(\pi_2^{-1}(\gamma))$ are smooth submanifolds of M for $\gamma \in \Gamma$, $\overline{M} \subset M$ is a immersed submanifold, and Φ is a top degree density on Γ. Then

$$\int_\Gamma \#(\overline{M} \cap M_\gamma)\Phi(\gamma) = \int_{\overline{M}} GT(\Phi).$$

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The Symplectic Structure on the Space of Geodesics

Consider the following diagram

$$\overline{Gr_1(\mathbb{R}^n)} \overset{p}{\leftarrow} S\mathbb{R}^n \overset{\varphi_F}{\overset{\simeq}{\rightarrow}} S^*\mathbb{R}^n \overset{i}{\hookrightarrow} T^*\mathbb{R}^n .$$

Consider the geodesic vector field $\bar{\mathcal{X}}(\bar{\xi}) := (\bar{\xi}, 0)$ for any $\bar{\xi} \in S\mathbb{R}^n$, and then φ_F induces another vector field $\mathcal{X} := d\varphi_F(\bar{\mathcal{X}})$,

$$\mathcal{X}(\xi) = (d\varphi_F(\bar{\mathcal{X}})(\varphi_F(\bar{\xi})) = (\bar{\xi}, 0)$$

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for $\xi = \varphi_F(\bar{\xi}) \in S^*\mathbb{R}^n$.

Lemma

$i_{\mathcal{X}}\omega = 0$ on $S^*\mathbb{R}^n$.

Then the Lie derivative of ω along the geodesic vector field $\mathcal{X}$,

$$\mathcal{L}_{\mathcal{X}}\omega = d i_{\mathcal{X}}\omega + i_{\mathcal{X}} d\omega = 0.$$

Lemma

$p_*(\bar{\mathcal{X}}) = 0$. i.e. $\bar{\mathcal{X}}$ is in the kernel of dp.

Theorem

There exists a symplectic form ω_0 on $\overline{Gr_1(\mathbb{R}^n)}$, such that $p^*\omega_0 = \bar{\omega} = (\varphi_F)^*\omega|_{S^*\mathbb{R}^n}$.

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Tangent Bundle of Unit Sphere

Consider the following diagram,

$$\begin{array}{ccccc} S\mathbb{R}^n & \overset{\varphi_F}{\overset{\simeq}{\longrightarrow}} & S^*\mathbb{R}^n & \overset{i}{\hookrightarrow} & T^*\mathbb{R}^n \\ \downarrow p & & & & \\ \overline{Gr_1(\mathbb{R}^n)} \overset{\psi}{\simeq} TS_F^{n-1} & \overset{\tilde{\varphi}_F}{\overset{\simeq}{\longrightarrow}} & T^*S_F^{n-1}. & & \end{array}$$

S_F^{n-1} as a Riemannian manifold carrying the metric $\langle \bar{u}, \bar{v} \rangle_{g_F} := \frac{\partial^2}{\partial s \partial t} F(\bar{\xi} + s\bar{u} + t\bar{v})|_{s=t=0}$ for any $\bar{u}, \bar{v} \in T_{\bar{\xi}} S_F^{n-1}$ has a natural symplectic structure induced from $T^*S_F^{n-1}$ by $\tilde{\varphi}_F : TS_F^{n-1} \to T^*S_F^{n-1}$, $\tilde{\varphi}_F(\bar{\eta}_{\bar{\xi}}) = \langle \bar{\eta}_{\bar{\xi}}, \cdot \rangle_{g_F}$.

Tangent Bundle of Unit Sphere

Theorem

ω_0 on $\overline{Gr_1(\mathbb{R}^n)} \overset{\psi}{\simeq} TS_F^{n-1}$ described in the following figure equals the symplectic form induced from the cotangent bundle $T^*S_F^{n-1}$ by $\tilde{\varphi}_F$.

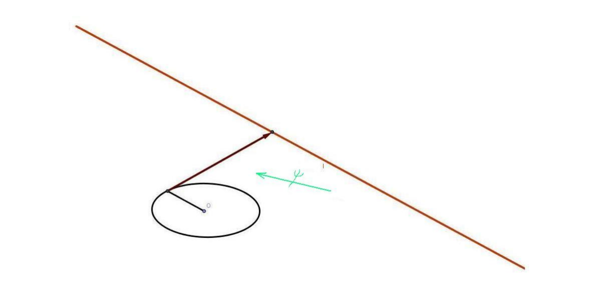

Figure: $\overline{Gr_1(\mathbb{R}^n)} \overset{\psi}{\simeq} TS_F^{n-1}$

Crofton Measure for the Length

Lemma

Suppose $\mathbf{L}(\overline{xy})$ *is the length of* $\overline{xy}$. *Then* $\int_c \alpha = \mathbf{L}(\overline{xy})$, *where* $c(t) := (x + \frac{t}{F(y-x)}(y-x), dF(\frac{y-x}{F(y-x)}))$, $t \in [0, F(y-x)]$, *and* $d\alpha = \omega$ *on* $S^*\mathbb{R}^n$.

Proposition

The Crofton measure on $\overline{Gr_1(\mathbb{R}^2)}$ *for the length is* $|\omega_0|$.

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Proof.

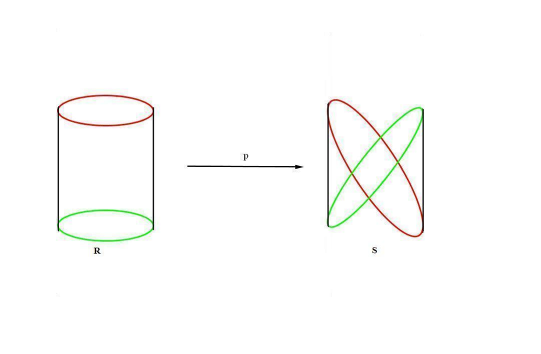

Figure: Map between cylinders of vectors and lines

Apply Stokes' theorem and the previous lemma, see the above figure,

$$\begin{aligned}\int_S |\omega_0| = \int_{p(R)} |\omega_0| = \int_R |p^*\omega_0| &= \int_R |\omega| \\ &= \int_{R^+} \omega + \int_{R^-} \omega \\ &= \int_{\partial R^+} \alpha + \int_{\partial R^-} \alpha \\ &= 4\mathbf{L}(\overline{xy}).\end{aligned}$$

Holmes-Thompson volumes

Definition

Let N be a k-dimensional Finsler manifold and D^*N be the codisc bundle of N , then the k-th Holmes-Thompson volume is defined as $vol_k(N) := \frac{1}{\epsilon_k}\int_{D^*N}|\omega^k|$, where ϵ_k is the Euclidean volume of k-dimensional Euclidean ball and ω is the canonical symplectic form on the cotangent bundle of N.

Lemma

$i^*\hat{\omega}_0 = \omega_0$ for $i : \overline{Gr_1(\Lambda)} \hookrightarrow \overline{Gr_1(\mathbb{R}^n)}$, where ω_0 and $\hat{\omega}_0$ are the natural symplectic forms on $\overline{Gr_1(\mathbb{R}^n)}$ and $\overline{Gr_1(\Lambda)}$ constructed in the way described in the second section.

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Volume of Hypersurfaces

Suppose N is a hypersurface in $(\mathbb{R}^n, F)$, then we have the following

Proposition

$vol_{n-1}(N) = \frac{1}{2\epsilon_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^n)}} \#(N \cap l)|\omega_0^{n-1}|$, *where* ω_0 *is the symplectic form on* $\overline{Gr_1(\mathbb{R}^n)}$.

This idea of intrinsic proof is given by Dr. Joseph H. G. Fu.
Let's consider the following diagram

$$S^*N \overset{\hat{i}}{\hookrightarrow} S^*\mathbb{R}^{n-1} \overset{\varphi_F^*}{\underset{\cong}{\rightarrow}} S\mathbb{R}^{n-1} \overset{i}{\hookrightarrow} S\mathbb{R}^n \overset{\varphi_F}{\underset{\cong}{\rightarrow}} S^*\mathbb{R}^n \overset{\pi}{\rightarrow} \overline{Gr_1(\mathbb{R}^n)}.$$

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(Diagram: $S^*N \overset{\hat{i}}{\hookrightarrow} S^*\mathbb{R}^{n-1} \overset{\varphi_{F^*}}{\overset{\cong}{\to}} S\mathbb{R}^{n-1} \overset{i}{\hookrightarrow} S\mathbb{R}^n \overset{\varphi_F}{\overset{\cong}{\to}} S^*\mathbb{R}^n \overset{\pi}{\to} \overline{Gr_1(\mathbb{R}^n)}$)

Proof.

(Outline) Using Stokes' theorem twice,

$$\begin{aligned}\int_{D^*N} \hat{\omega}^{n-1} &= \int_{\partial(D^*N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} = \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2} \\ &\quad + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} \\ &= \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2},\end{aligned}$$

$$\begin{aligned}\int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{\partial(S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N))} \alpha \wedge \omega^{n-2} \\ &= \int_{S^*N} \hat{i}^* j^* \alpha \wedge \hat{i}^* j^* \omega^{n-2} \\ &\quad + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{i}^* j^* \alpha \wedge \hat{i}^* j^* \omega^{n-2} \\ &= \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2}.\end{aligned}$$

Transform the integral to $\overline{Gr_1(\mathbb{R}^n)}$,

$$\begin{aligned}\int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \pi^* \omega_0^{n-1} \\ &= \int_{\pi^{-1}(l) \in S_+^*\mathbb{R}^n \cap \pi_0^{-1}(N)} \#(N \cap l) \omega_0^{n-1} \\ &= \int_{l \in \overline{Gr_1^+(\mathbb{R}^n)}} \#(N \cap l) \omega_0^{n-1}.\end{aligned}$$

□

Gelfand Transform

From spherical harmonics,

Fact

There exists an even function f on S^{n-1}, such that

$$L(\overline{xy}) = \frac{1}{4}\int_{\xi\in S^{n-1}} |\langle \xi, \overrightarrow{xy}\rangle| f(\xi)\Omega,$$

where Ω is the standard volume form on S^{n-1}.

Proposition

$GT(|f\Omega\wedge dr|) = |\omega_0|$, where GT is the Gelfand transform for the double fibraton $\overline{Gr_1(\mathbb{R}^n)} \overset{\pi_1}{\leftarrow} \mathcal{I} \overset{\pi_2}{\rightarrow} \overline{Gr_{n-1}(\mathbb{R}^n)}$ and is r the Euclidean distance of a hyperplane H to the origin.

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Proof.

It is sufficient to show the equality claimed holds for any two tangent vectors of $\overline{Gr_1(\mathbb{R}^n)}$.

First, for any plane $\Pi \subset \mathbb{R}^n$, by the previous fact and the fundamental theorem of Gelfand transform, we show that

$$\int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l)|GT(f\Omega \wedge dr)| = \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l)|\omega_0|,$$

which implies $GT(|f\Omega \wedge dr|)|_{\overline{Gr_1(\Pi)}} = |\omega_0|_{\overline{Gr_1(\Pi)}}$ by the injectivity of cosine transform.

Next, we define a natural basis for the tangent space of $\overline{Gr_1(\mathbb{R}^n)}$ and analyse four cases in terms of the properties of the two tangent vector to be chosen, showing that the equality holds for each of the four cases, in three of them both sides of the equality are actually 0.

□

For k-th Holmes-Thompson Volume

Let $\Omega_{n-1} := f\Omega \wedge dr$ and define a map

$$\pi : \overline{Gr_{n-1}(\mathbb{R}^n)}^k \backslash \triangle_k \to \overline{Gr_{n-k}(\mathbb{R}^n)}$$
$$\pi((H_1, \cdots, H_k)) = H_1 \cap \cdots \cap H_k,$$

where $\triangle_k = \{(H_1, \cdots, H_k) : dim(H_1 \cap \cdots \cap H_k) > n - k\}$.
Then define

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Then define

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Consider the double fibration, $\overline{Gr_1(\mathbb{R}^n)} \overset{\pi_{1,k}}{\longleftarrow} \mathcal{I}_k \overset{\pi_{2,k}}{\longrightarrow} \overline{Gr_{n-k}(\mathbb{R}^n)}$, where $\mathcal{I}_k = \left\{(l, S) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-k}(\mathbb{R}^n)} : l \subset S\right\}$, and the following diagram

$$\begin{array}{ccccc} \overline{Gr_1(\mathbb{R}^n)} & \overset{\pi_{1,k}}{\longleftarrow} & \mathcal{I}_k & \overset{\pi_{2,k}}{\longrightarrow} & \overline{Gr_{n-k}(\mathbb{R}^n)} \\ & \tilde{\pi}_1 \nwarrow & \uparrow \tilde{\pi} & & \uparrow \pi \\ & & \mathcal{H} & \overset{\tilde{\pi}_2}{\longrightarrow} & \overline{Gr_{n-1}(\mathbb{R}^n)}^k, \end{array}$$

where $\mathcal{H} :=$ $\left\{(l, (H_1, H_2, \cdots, H_k)) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-1}(\mathbb{R}^n)}^k : l \subset H_1 \cap \cdots \cap H_k\right\}$.

Proposition

$GT(|\Omega_{n-k}|) = |\omega_0^k|$.

Fix $S \in \overline{Gr_{k+1}(\mathbb{R}^n)}$, and define a map by taking an intersection

$$\pi_S : \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \triangle(S) \to \overline{Gr_1(S)}$$
$$\pi_S(H^{n-k}) = H^{n-k} \cap S$$

for $H^{n-k} \in \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \triangle(S)$.

Proposition

$(\pi_S)_*|\Omega_{n-k}| = |\omega_0^k|_{\overline{Gr_1(S)}}$.

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Crofton Measures for H-T Volumes

Theorem

(Alvarez) Suppose N is a k-dimensional submanifold in $(\mathbb{R}^n, F)$. Then $vol_k(N) = \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P) |\Omega_{n-k}|$.

Proof.

By the previous proposition and the theorem on hypersurface,

$$\begin{aligned} vol_k(N) &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l) |\omega_0^k| \\ &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l) (\pi_S)_* |\Omega_{n-k}| \\ &= \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P) |\Omega_{n-k}|. \end{aligned}$$

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Crofton Measures for H-T Volumes

Theorem

(Alvarez) Suppose N is a k-dimensional submanifold in $(\mathbb{R}^n, F)$. Then $vol_k(N) = \frac{1}{2\epsilon_k}\int_{P\in\overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N\cap P)|\Omega_{n-k}|$.

Proof.

By the previous proposition and the theorem on hypersurface,

$$\begin{aligned} vol_k(N) &= \frac{1}{2\epsilon_k}\int_{l\in\overline{Gr_1(S)}} \#(N\cap l)|\omega_0^k| \\ &= \frac{1}{2\epsilon_k}\int_{l\in\overline{Gr_1(S)}} \#(N\cap l)(\pi_S)_*|\Omega_{n-k}| \\ &= \frac{1}{2\epsilon_k}\int_{P\in\overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N\cap P)|\Omega_{n-k}|. \end{aligned}$$

□

Holmes-Thompson Valuations

Because of the Crofton measures for Holmes-Thompson Volumes, one can extend these volumes to valuations

$$\mu_k(K) := \int_{H \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(K \cap H)\phi_k(H),$$

for any compact convex subset K of the Minkowski space $(\mathbb{R}^n, F)$, where $\phi_k := |\Omega_{n-k}|$ is the Crofton measures for k-th Holmes-Thompson volume vol_k.

Graded Ring Structure

Let $\pi_{k,l} : \overline{Gr_{n-k}(\mathbb{R}^n)} \times \overline{Gr_{n-l}(\mathbb{R}^n)} \setminus \Delta \to \overline{Gr_{n-k-l}(\mathbb{R}^n)}$ be taking the intersection, then $\pi_{k,l*}(\phi_k \times \phi_l) = \phi_{k+l}$, which implies $\mu_k \cdot \mu_l = \mu_{k+l}$.

Definition

Let $\mu_A(K) := vol(K + A)$ and $\mu_B(K) := vol(K + B)$, for any convex bodies A and B with strictly convex smooth boundaries in $(\mathbb{R}^n, F)$, then the Alesker product of μ_A and μ_B is defined as $\mu_A \cdot \mu_B(K) := vol \times vol(\Delta(K) + A \times B)$, where $\Delta(K)$ is the diagonal embedding of K in $\mathbb{R}^n \times \mathbb{R}^n$, and the Alesker product of μ_k and μ_l, $\mu_k \cdot \mu_l$, is defined by the linear span of the above products.

Theorem

(Alesker-Bernig) The space of valuations generated by Holmes-Thompson valuations is a graded ring w. r. t. Alesker product.