Symmetry Types of Links with 2 Components

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1 Introduction

Take a link L with 2 components K_1 and K_2, then the symmetry type of L can be any subgroup of Γ_2 of a semidirect product of $\mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ with 16 elements.

2 Extremal Symmetries

First we know that Γ_2 has trivial subgroups $\{e\}$ and Γ_2 itself. By analogue to the case of knots, we can say the followings:

(I) If the symmetry group of L is $\{e\}$, we call L has "None" symmetry. The example for this will be a a link of two different knots with "None" symmetry, like 9_{32} linked with another "None" symmetry knot.

(II) If the symmetry group of L is Γ_2, we call L has "Full" symmetry. For example, 4_1 linked with 4_1.

3 Canonical

Secondly, considering the canonical normal subgroup $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ of Γ_2, we know that the link has full symmetry on each component, but does not have exchanging symmetry. For instance, a link with 4_1 and another full symmetry knot. We give a name for this type of symmetry,

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say "Normal" symmetry associated to the corresponding normal subgroup.

In addition, the second component $\mathbb{Z}_2 = \langle(1,1,1,(1,2))\rangle$ of the semidirect product of Γ_2 is a subgroup of Γ_2. It corresponds to links with exchanging symmetry, but no other symmetries, like mirroring or inverting, on each component knot. The example for this case is 9_{32} linked with 9_{32}. Since this kind of links have only exchanging symmetry, we can call it "Exchangeable" symmetry.

4 Derived from Hopf Links

Now let's look at the case of Hopf link 2_1^2 that Jason did in [1], the symmetry group of the Hopf link is generated by $\alpha = (-1,1,-1,(1,2))$ and $\gamma = (1,1,1,(1,2))$, Jason computed the whole symmetry group, which indeed is

$$\left\{e, \alpha, \alpha^2, \alpha^3, \gamma, \alpha\gamma, \alpha^2\gamma, \alpha^3\gamma\right\} = \left\{e, \alpha, \alpha^2, \alpha^3\right\} \rtimes \{e, \gamma\},$$

in which the relation is $\{\alpha^4 = \gamma^2 = e, \alpha\gamma = \gamma^3\alpha\}$. This is a dihedral group, as well as the symmetry group of a square with respect to rigid motions. Thus for links with this type of symmetry, we can give a name "Dihedral" symmetry.

Next, the group $\{e, \alpha, \alpha^2, \alpha^3\}$ is a normal subgroup of the "Dihedral" symmetry group. Since γ is not in this group and it is a cyclic subgroup, we'd like to give a name "Cyclic-nonexchangeable" symmetry corresponding to the group

$$\{e, (-1,1,-1,(1,2)), (1,-1,-1,e), (-1,-1,1,(1,2))\}.$$

5 Symmetry Types Inherited from Knots

For the non-exchangeable cases, in terms of the "mirror" and "reversible" symmetry type of knots, we have the following subcases:

(I) If L is a link of two different knots K_1 and K_2 with "mirror" symmetry, and K_1 and K_2 are not interchangeable, then the corresponding symmetry group is

$$\mathbb{Z}_2 = \{e, (-1,1,1,e)\}.$$

Using the name "mirror" in symmetry type of knots, we call "Mirrorable" for links here.

(II) If L is a link of two different knots K_1 and K_2 with "reversible" symmetry, K_1 and K_2 are not interchangeable, and just reversing a single component K_1 or K_2 will produce a different link, then the corresponding symmetry group is

$$\mathbb{Z}_2 = \{e, (1, -1, -1, e)\}.$$

Using the name "reversible" in symmetry type of knots, we call "Reversible" for links here.

(III) If L is a link of two different knots K_1 and K_2 with "amphicharal" symmetry, and K_1 and K_2 are not interchangeable, then the corresponding symmetry group is

$$\mathbb{Z}_2 = \{e, (-1, -1, -1, e)\}.$$

Using the name "amphicharal" in symmetry type of knots, we call "Amphicharal" for links here.

6 Symmetry Types with Subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Now that in the first three components of the elements in the group constitutes the group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle(-1, 1, 1, e)\rangle * \langle(1, -1, 1, e)\rangle * \langle(1, 1, -1, e)\rangle,$$

then we have other cases of subgroups

$$\{e, (1, -1, 1, e)\}, \tag{1}$$

$$\{e, (1, 1, -1, e)\}, \tag{2}$$

$$\{e, (-1, -1, 1, e)\}, \tag{3}$$

$$\{e, (-1, 1, -1, e)\}, \tag{4}$$

and

$$\{e, (-1, 1, -1, e), (-1, -1, 1, e), (1, -1, -1, e)\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{5}$$

for symmetry types of links than the subgroups

$$\begin{gathered}\{e\}\,,\\ \{e,(-1,1,1,e)\}\,,\\ \{e,(1,-1,-1,e)\}\,,\\ \{e,(-1,-1,-1,e)\}\,,\end{gathered}$$

and the whole group

$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2,$$

which have been listed in the above cases. For those cases, we'd like to give the names "Levo-reversible" for the symmetry type corresponding to (1), "Dexter-reversible" to (2), "Amphicharal-levo-reversible" to (3), and "Amphicharal-dexter-reversible" to (4). Moreover, the symmetry type of (5) can be named "Amphicharal-dual-reversible".

7 Composed by "Exchangeable" Symmetry

The generating symmetry types by "Exchangeable" symmetry case and non-exchangeable cases above will be the following:

(I) If L is a link of two copies of knots with "mirror" symmetry, then the corresponding symmetry group is

$$\begin{aligned}\mathbb{Z}_2\times\mathbb{Z}_2 &= \{e,(-1,1,1,e)\}*\{e,(1,1,1,(1,2))\}\\ &= \langle(-1,1,1,e),(1,1,1,(1,2))\rangle\,.\end{aligned}$$

Using the name "mirror" in symmetry type of knots, we call "Mirrorable-exchangeable" for links here.

(II) If L is a link of two copies of knots with "reversible" symmetry, then the corresponding symmetry group is

$$\begin{aligned}\mathbb{Z}_2\times\mathbb{Z}_2 &= \{e,(1,-1,-1,e)\}*\{e,(1,1,1,(1,2))\}\\ &= \langle(1,-1,-1,e),(1,1,1,(1,2))\rangle\,.\end{aligned}$$

Using the name "reversible" in symmetry type of knots, we call "Reversible-exchangeable" for links here.

(III) If L is a link of two copies of knots with "amphicharal" symmetry, then the corresponding symmetry group is

$$\begin{aligned}\mathbb{Z}_2 \times \mathbb{Z}_2 &= \{e, (-1, -1, -1, e)\} * \{e, (1, 1, 1, (1, 2))\} \\ &= \langle (-1, -1, -1, e), (1, 1, 1, (1, 2)) \rangle .\end{aligned}$$

Using the name "amphicharal" in symmetry type of knots, we call "Amphicharal-exchangeable" for links here.

Also, we have the other five symmetry types corresponding to subgroups (1), (2), (3), (4), and (5) multiplied by the exchanging subgroup $\{e, (1, 1, 1, (1, 2))\}$. Therefore, we have the following symmetry types with respect to the subgroups:

(I) Multiplying (1) by $\{e, (1, 1, 1, (1, 2))\}$, we get a group H which is a subgroup of Γ_2

$$H = \{e, (1, -1, 1, e)\} * \{e, (1, 1, 1, (1, 2))\} .$$

Since both $\{e, (1, -1, 1, e)\}$ and $\{e, (1, 1, 1, (1, 2))\}$ are index-2 subgroups of H, they are actually normal subgroups of H. Therefore, the product in fact is a direct product. Thus

$$\begin{aligned}H &= \{e, (1, -1, 1, e)\} \times \{e, (1, 1, 1, (1, 2))\} \\ &= \mathbb{Z}_2 \times \mathbb{Z}_2.\end{aligned}$$

For this case we give a name "Levo-reversible-exchangeable" for this symmetry type.
Similarly, we have

(II) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, (1, 1, -1, e)\} \times \{e, (1, 1, 1, (1, 2))\}$ with a name "Dexter-reversible-exchangeable" for the symmetry type.

(III) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, (-1, -1, 1, e)\} \times \{e, (1, 1, 1, (1, 2))\}$ with a name "Amphicharal-levo-reversible-exchangeable" for the symmetry type.

(IV) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, (-1, 1, -1, e)\} \times \{e, (1, 1, 1, (1, 2))\}$ with a name "Amphicharal-dexter-reversible-exchangeable" for the symmetry type.

(V) Since every element in (5) is commutable with $(1, 1, 1, (1, 2))$, the product is indeed a direct product too. Therefore we have the symmetry group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, (-1, 1, -1, e), (-1, -1, 1, e), (1, -1, -1, e)\} \times \{e, (1, 1, 1, (1, 2))\}$$

with a name “Amphicharal-dual-reversible-exchangeable” for the symmetry type.

8 Conclusions

In summary, we have the symmetry types “None”, “Full”, “Normal”, “Dihedral”, “Cyclic-nonexchangeable”, “Exchangeable”, “Mirrorable”, “Reversible”, “Amphicharal”, “Levo-reversible”, “Dexter-reversible”, ”Amphicharal-levo-reversible”, “Amphicharal-dexter-reversible”, “Amphicharal-dual-reversible”, “Mirrorable-exchangeable”, “Reversible-exchangeable”, “Amphicharal-exchangeable”, “Levo-reversible-exchangeable”, “Dexter-reversible-exchangeable”, “Amphicharal-levo-reversible-exchangeable”, “Amphicharal-dexter-reversible-exchangeable”, and “Amphicharal-dual-reversible-exchangeable” for links with 2 components.

References

[1] Jason Catarella, *Notes on link tabulations*

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