

SYMPLECTIC STRUCTURE ON THE SPACE OF AFFINE LINES

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The symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$ of affine lines in Euclidean space is constructed.

1. NATURAL TWO FORM ON $S\mathbb{R}^n$.

Let $T\mathbb{R}^n = \{(x, \xi) : x \in \mathbb{R}^n, \xi \in T_x\mathbb{R}^n\}$, the natural symplectic form

$$(1.1) \quad d\alpha = \sum_{i=1}^n dx_i \wedge d\xi_i$$

on $T\mathbb{R}^n$ is induced from the natural symplectic structure on $T^*\mathbb{R}^2$ and the Euclidean metric. Then for the sphere-bundle $S\mathbb{R}^n = \{(x, \xi) : x \in \mathbb{R}^n, \langle \xi, \xi \rangle = 1\}$, the inclusion $i : S\mathbb{R}^n \rightarrow T\mathbb{R}^n$ induces a two form on $S\mathbb{R}^n$, which is

$$(1.2) \quad \bar{\omega} := i^*d\alpha = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Now let us consider a flow on $S\mathbb{R}^n$,

$$(1.3) \quad \begin{aligned} F_t : S\mathbb{R}^n &\rightarrow S\mathbb{R}^n \\ F_t((x, \xi)) &= (x + t\xi, \xi), \end{aligned}$$

then F_t induces a pushforward $(F_{t*})_{(x, \xi)}$ from $T_{(x, \xi)}S\mathbb{R}^n$ to $T_{(x+t\xi, \xi)}S\mathbb{R}^n$. Furthermore we have the following proposition.

Proposition 1.1. $F_t^*\bar{\omega} = \bar{\omega}$.

Proof. A direct computation on differential forms gives

$$\begin{aligned} F_t^*\bar{\omega} &= F_t^*\left(\sum_{i=1}^n dx_i \wedge d\xi_i\right) \\ &= \sum_{i=1}^n d(x_i + t\xi_i) \wedge d\xi_i \\ &= \sum_{i=1}^n dx_i \wedge d\xi_i \\ &= \bar{\omega}. \end{aligned}$$

□

2. PUSHFORWARD UNDER PROJECTION

Let $\overline{Gr_1(\mathbb{R}^n)}$ be the space of affine lines in \mathbb{R}^n , then for any $l \in \overline{Gr_1(\mathbb{R}^n)}$, there is a unique direction vector ξ , and a hyperplane H which is normal to l , let $x = H \cap l$. Conversely, (x, ξ) satisfying $\langle x, \xi \rangle = 0$, $\langle \xi, \xi \rangle = 1$ determines a unique line l . Therefore, $\overline{Gr_1(\mathbb{R}^n)} = \{(x, \xi) : \langle x, \xi \rangle = 0, \langle \xi, \xi \rangle = 1\}$, which is homeomorphic to $TS^{n-1} \subset T\mathbb{R}^n$. The tangent bundle TS^{n-1} is a $2n - 2$ dimensional submanifold of $T\mathbb{R}^n$, which gives a manifold structure on $\overline{Gr_1(\mathbb{R}^n)}$.

Now let us consider the natural projection

$$(2.1) \quad \begin{aligned} \pi : S\mathbb{R}^n &\rightarrow \overline{Gr_1(\mathbb{R}^n)} \\ \pi((x, \xi)) &= (x - \langle x, \xi \rangle \xi, \xi). \end{aligned}$$

We have the following proposition.

Proposition 2.1. $ker(\pi_*)_{(x, \xi)} = \{k \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i} : k \in \mathbb{R}\}$ for $(x, \xi) \in S\mathbb{R}^n$.

Proof. First we have $dim(ker(\pi_*)) = 1$ since the rank of the Jacobi matrix of π retracted to x component

$$\begin{pmatrix} 1 - \xi_1^2 & -\xi_1 \xi_2 & \cdot & -\xi_1 \xi_{n-1} \\ -\xi_2 \xi_1 & 1 - \xi_2^2 & \cdot & -\xi_2 \xi_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ -\xi_{n-1} \xi_1 & -\xi_{n-1} \xi_2 & \cdot & 1 - \xi_{n-1}^2 \\ -\xi_n \xi_1 & -\xi_n \xi_2 & \cdot & -\xi_n \xi_{n-1} \end{pmatrix}$$

is $n - 1$.

Since $\sum_{i=1}^n \xi_i^2 = 1$, then there exists some i_0 , such that $\xi_{i_0} \neq 0$. Without loss of generality, we assume $\xi_n \neq 0$. Dropping x_n , we get a local coordinates, therefore

$$\begin{aligned} \pi_* \left(\sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i} \right) &= \sum_{i=1}^{n-1} \xi_i \frac{\partial}{\partial x^i} - \sum_{i=1}^{n-1} \sum_{j=1}^n \xi_j \xi_i \xi_j \frac{\partial}{\partial x^i} \\ &= \sum_{i=1}^{n-1} \xi_i \left(1 - \sum_{j=1}^n \xi_j^2 \right) \frac{\partial}{\partial x^i} \\ &= 0. \end{aligned}$$

Hence

$$ker(\pi_*) = \left\{ k \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i} : k \in \mathbb{R} \right\}.$$

□

Proposition 2.2. $\bar{\omega}(v, \cdot) = 0$ for $v \in ker(\pi_*)$.

Proof. By Proposition 3.2, $v = k \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i}$ for some $k \in \mathbb{R}$, then we have

$$\begin{aligned} \bar{\omega}(v, \cdot) &= \sum_{i=1}^n dx_i \wedge d\xi_i \left(k \sum_{i=1}^n \xi_i \frac{\partial}{\partial x^i}, \cdot \right) \\ &= k \sum_{i=1}^n \xi_i d\xi_i \\ &= 0, \end{aligned}$$

since $\sum_{i=1}^n \xi_i^2 = 1$.

□

Furthermore, we have a remark

Remark 2.3. If $(\pi_*)_{(x,\xi)}(v) = (\pi_*)_{(x,\xi)}(\bar{v})$ and $(\pi_*)_{(x,\xi)}(w) = (\pi_*)_{(x,\xi)}(\bar{w})$, then there exist some $v_0 \in \ker(\pi_*)$ and $w_0 \in \ker(\pi_*)$, such that $\bar{v} = v + v_0$ and $\bar{w} = w + w_0$. Therefore, $\bar{\omega}(\bar{v}, \bar{w}) = \bar{\omega}(v + v_0, w + w_0) = \bar{\omega}(v, w)$ by Proposition 2.2.

3. SYMPLECTIC STRUCTURE ON $\overline{Gr_1(\mathbb{R}^n)}$

Now we can construct a symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$ of affine lines in \mathbb{R}^n . By Proposition 1.1 and Remark 2.4, $\bar{\omega}_{(x,\xi)}(v, w)$ is independent of the choices of preimages under the pushforward induced by the natural projection. Thus we have a well-defined two form ω on $\overline{Gr_1(\mathbb{R}^n)}$,

$$(3.1) \quad \omega_{\pi(x,\xi)}(\tilde{v}, \tilde{w}) := \bar{\omega}_{(x,\xi)}(v, w),$$

where $(\pi_*)_{(x,\xi)}(v) = \tilde{v}$, $(\pi_*)_{(x,\xi)}(w) = \tilde{w}$. Moreover, ω gives rise to a symplectic structure on $\overline{Gr_1(\mathbb{R}^n)}$, and in fact it turns out to be natural.

Theorem 3.1. The symplectic form ω on $\overline{Gr_1(\mathbb{R}^n)}$ is the canonical symplectic form on $TS^{n-1} \cong \overline{Gr_1(\mathbb{R}^n)}$.

Proof. Let $\tilde{\omega} = \sum_{i=1}^n d\bar{x}_i \wedge d\bar{\xi}_i$ be the canonical symplectic form on

$$TS^{n-1} = \{(\bar{x}, \bar{\xi}) : \langle \bar{x}, \bar{\xi} \rangle = 0, \langle \bar{\xi}, \bar{\xi} \rangle = 1\}.$$

then

$$\begin{aligned} \pi^* \tilde{\omega} &= \sum_{i=1}^n d\bar{x}_i \wedge d\bar{\xi}_i \\ &= \sum_{i=1}^n d(x_i - \langle x, \xi \rangle \xi_i) \wedge d\xi_i \\ &= \sum_{i=1}^n dx_i \wedge d\xi_i - \sum_{i=1}^n \langle x, \xi \rangle d\xi_i \wedge d\xi_i - \sum_{i=1}^n d\langle x, \xi \rangle \wedge \xi_i d\xi_i \\ &= \sum_{i=1}^n dx_i \wedge d\xi_i \end{aligned}$$

since $\sum_{i=1}^n \xi_i^2 = 1$.

Then by (3.1) the conclusion follows. \square