TANGENT BUNDLE OF UNIT SPHERE IN MINKOWSKI SPACE AND SYMPLECTIC STRUCTURE

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Abstract. We show that the space of oriented lines and the tangent bundle of unit sphere in Minkowski space are symplectomorphic.

Let us consider a Minkowski plane $(\mathbb{R}^2, F)$ first, where F is a Finsler metric. The natural symplectic form on $T^*\mathbb{R}^2$ is $dx \wedge d\bar{\xi} + dy \wedge d\bar{\eta}$, and then the natural symplectic form on $T\mathbb{R}^2$ induce by the Finsler metric F is

$$\begin{aligned} \omega : &= dx \wedge d\frac{\partial F}{\partial \xi} + dy \wedge d\frac{\partial F}{\partial \eta} \\ &= \frac{\partial^2 F}{\partial \xi^2} dx \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta}(dx \wedge d\eta + dy \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} dy \wedge d\eta. \end{aligned}$$

Define a projection $\pi : T\mathbb{R}^2 \to \overline{Gr_1(\mathbb{R}^2)}$ by

$$\pi((x,y);(\xi,\eta)) = ((x,y) - dF(\xi,\eta)((x,y))(\xi,\eta);(\xi,\eta)). \tag{0.1}$$

Let S_F be the unit circle in the Minkowski plane and TS_F be its tangent bundle. It is a fact that $TS_F \cong \overline{Gr_1(\mathbb{R}^2)}$. On the other hand, since TS_F is embedded in $T\mathbb{R}^2$, it inherits a natural symplectic form $\omega_0 := \omega|_{TS_F}$ from $T\mathbb{R}^2$.

Theorem 0.1. $\pi^*\omega_0 = \omega|_{S^*\mathbb{R}^2}$.

Proof. Applying the equality

$$\frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta = 0, \tag{0.2}$$

we obtain

$$\begin{aligned} \pi^*\omega_0 &= \frac{\partial^2 F}{\partial \xi^2} d(x - dF(\xi,\eta)((x,y))\xi) \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta}(d(x - dF(\xi,\eta)((x,y))\xi) \wedge d\eta \\ &\quad + d(y - dF(\xi,\eta)((x,y))\eta) \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} d(y - dF(\xi,\eta)((x,y))\eta) \wedge d\eta \\ &= \frac{\partial^2 F}{\partial \xi^2} dx \wedge d\xi + \frac{\partial^2 F}{\partial \xi \partial \eta}(dx \wedge d\eta + dy \wedge d\xi) + \frac{\partial^2 F}{\partial \eta^2} dy \wedge d\eta \\ &\quad - d(dF(\xi,\eta)((x,y))) \wedge (\frac{\partial^2 F}{\partial \xi^2}\xi d\xi + \frac{\partial^2 F}{\partial \eta^2}\eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi d\eta + \eta d\xi)). \end{aligned}$$

By the positive homogeneity of F, one can get the useful fact that $F(\xi,\eta) = \xi\frac{\partial F}{\partial \xi} + \eta\frac{\partial F}{\partial \eta}$. Therefore,

$$\xi\frac{\partial F}{\partial \xi} + \eta\frac{\partial F}{\partial \eta} = 1. \tag{0.3}$$

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By differentiating (0.3), we get

$$\frac{\partial^2 F}{\partial \xi^2}\xi d\xi + \frac{\partial^2 F}{\partial \eta^2}\eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi d\eta + \eta d\xi) + \frac{\partial F}{\partial \xi}d\xi + \frac{\partial F}{\partial \eta}d\eta = 0. \tag{0.4}$$

Applying (0.2) again, we have

$$\frac{\partial^2 F}{\partial \xi^2}\xi d\xi + \frac{\partial^2 F}{\partial \eta^2}\eta d\eta + \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi d\eta + \eta d\xi) = 0. \tag{0.5}$$

Thus the claim follows. □

Remark 0.2. For a n-dimensional Minkowski space $(\mathbb{R}^n, F)$, we just need to add more indices, then the theorem above is also true for $(\mathbb{R}^n, F)$.

Therefore, letting F be a Finsler metric on $\mathbb{R}^n$ and S_F be the unit sphere in the Minkowski space $(\mathbb{R}^n, F)$, we obtain a general theorem

Theorem 0.3. *The symplectic form on the space of lines in a Minkowski space $(\mathbb{R}^n, F)$ is the canonical symplectic form on the tangent bundle TS_F as imbedded in $T\mathbb{R}^n$.*

Remark 0.4. Theorem 0.3 gives us a perspective that we can transform calculus on $\overline{Gr_1(\mathbb{R}^2)}$ to ones on TS_F.

Another remark from the proof of Theorem 0.2 is

Remark 0.5. A combination of (0.5) and Gelfand transform (see [1]) may give us a short proof of the general Crofton formula for Minkowski space.

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References

[1] Integral Geometry on Minkowski p-Space