The Symmetry Group of Composite Link $2_1^2 m \sharp 2_1^2$

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1 Introduction

We show that the symmetry group of the composite link $2_1^2 m \sharp 2_1^2$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \alpha\} \times \{1, \beta\} \times \{1, \gamma\}$, where $\alpha = (1, 1, -1, -1, (2, 3))$, $\beta = (1, -1, -1, -1, e)$, and $\gamma = (-1, -1, 1, 1, e)$.

2 Procedures

Following Jason's algorithm, we compute the compatible (p,r) permutations first. Since the compatible permutations just determine which component of each link is connected, the compatible permutations are the same with the case $2_1^2 \sharp 2_1^2$ that Jason computed. So the compatible permutations are $p = (2, 3), r = (1, 2)$ and $p = e, r = e$.

Now, let's go to the next step, finding $\bar{p_1}$ and $\bar{p_2}$. It is the same process with Jason's notes([1]). Therefore, $\bar{p_1} = \bar{p_2} = e$.

Continuing to follow the example on Jason's note and using the fact that $2_1^2 m$ and 2_1^2 are in different cosets and 2_1^2 has symmetry group $\langle (1, -1, -1, e), (-1, 1, -1, e), (1, 1, 1, (1, 2) \rangle$, we have 16 cases as the following:

1. $\gamma = (1, 1, 1, 1, (2, 3)); \gamma_1 = (1, 1, 1, e); \gamma_2 = (1, 1, 1, e)$.

   Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

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2. $\gamma = (1, 1, 1, -1, (2, 3)); \gamma_1 = (1, 1, 1, e); \gamma_2 = (1, 1, -1, e)$.

   Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

3. $\gamma = (1, 1, -1, 1, (2, 3)); \gamma_1 = (1, 1, -1, e); \gamma_2 = (1, 1, 1, e)$.

   Since $L_2^{\gamma_1}$ is in the coset of L_1, but $L_1^{\gamma_2}$ is not in the coset of L_2, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

4. $\gamma = (1, 1, -1, -1, (2, 3)); \gamma_1 = (1, 1, -1, e); \gamma_2 = (1, 1, -1, e)$.

   Since $L_2^{\gamma_1}$ is in the coset of L_1, and $L_1^{\gamma_2}$ is in the coset of L_2, this case IS in the symmetry group of $2_1^2 m \sharp 2_1^2$.

5. $\gamma = (1, -1, 1, 1, (2, 3)); \gamma_1 = (1, -1, 1, e); \gamma_2 = (1, -1, 1, e)$.

   Since $L_2^{\gamma_1}$ is in the coset of L_1, and $L_1^{\gamma_2}$ is in the coset of L_2, this case IS in the symmetry group of $2_1^2 m \sharp 2_1^2$.

6. $\gamma = (1, -1, 1, -1, (2, 3)); \gamma_1 = (1, -1, 1, e); \gamma_2 = (1, -1, -1, e)$.

   Since $L_1^{\gamma_2}$ is not in the coset of L_2, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

7. $\gamma = (1, -1, -1, 1, (2, 3)); \gamma_1 = (1, -1, -1, e); \gamma_2 = (1, -1, 1, e)$.

   Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

8. $\gamma = (1, -1, -1, -1, (2, 3)); \gamma_1 = (1, -1, -1, e); \gamma_2 = (1, -1, -1, e)$.

   Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

The next 8 cases of mirror image are the following:

9. $\gamma = (-1, 1, 1, 1, (2, 3)); \gamma_1 = (-1, 1, 1, e); \gamma_2 = (-1, 1, 1, e)$.

   Since $L_2^{\gamma_1}$ is in the coset of L_1, and and $L_1^{\gamma_2}$ is in the coset of L_2, this case IS in the symmetry group of $2_1^2 m \sharp 2_1^2$.

10. $\gamma = (-1, 1, 1, -1, (2, 3)); \gamma_1 = (-1, 1, 1, e); \gamma_2 = (-1, 1, -1, e)$.

    Since $L_1^{\gamma_2}$ is not in the coset of L_2, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

11. $\gamma = (-1, 1, -1, 1, (2,3)); \gamma_1 = (-1, 1, -1, e); \gamma_2 = (-1, 1, 1, e)$.

    Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

12. $\gamma = (-1, 1, -1, -1, (2,3)); \gamma_1 = (-1, 1, -1, e); \gamma_2 = (-1, 1, -1, e)$.

    Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

13. $\gamma = (-1, -1, 1, 1, (2,3)); \gamma_1 = (-1, -1, 1, e); \gamma_2 = (-1, -1, 1, e)$.

    Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

14. $\gamma = (-1, -1, 1, -1, (2,3)); \gamma_1 = (-1, -1, 1, e); \gamma_2 = (-1, -1, -1, e)$.

    Since $L_2^{\gamma_1}$ is not in the coset of L_1, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

15. $\gamma = (-1, -1, -1, 1, (2,3)); \gamma_1 = (-1, -1, -1, e); \gamma_2 = (-1, -1, 1, e)$.

    Since $L_1^{\gamma_2}$ is not in the coset of L_2, this case is NOT in the symmetry group of $2_1^2 m \sharp 2_1^2$.

16. $\gamma = (-1, -1, -1, -1, (2,3)); \gamma_1 = (-1, -1, -1, e); \gamma_2 = (-1, -1, -1, e)$.

    Since $L_2^{\gamma_1}$ is in the coset of L_1, and and $L_1^{\gamma_2}$ is in the coset of L_2, this case IS in the symmetry group of $2_1^2 m \sharp 2_1^2$.

In summary, the set of elements involving $(2,3)$ as the permutation in the symmetry group is

$$S_1 = \{(1, 1, -1, -1, (2,3)), (1, -1, 1, 1, (2,3)),\\ (-1, 1, 1, 1, (2,3)), (-1, -1, -1, -1, (2,3)).$$

For another compatible permutation $p = r = e$, we compare $L_1^{\gamma_1}$ and L_1, $L_2^{\gamma_2}$ and L_2 as Jason did. We'll have the following set of elements in the symmetry group:

$$S_2 = \{(1, 1, 1, 1, e), (1, -1, -1, -1, e),\\ (-1, -1, 1, 1, e), (-1, 1, -1, -1, e).$$

3 Structure Analysis

Now we are going to analyze the structure of the symmetry group with the multiplication operation defined in Jason's notes. Let $\alpha = (1, 1, -1, -1, (2, 3))$, $\beta = (1, -1, -1, -1, e)$, $\gamma = (-1, -1, 1, 1, e)$, $\delta = (-1, 1, -1, -1, e)$, and the unit element $1 = (1, 1, 1, 1, e)$, we have $(1, -1, 1, 1, (2, 3)) = \alpha\beta$, $(-1, 1, 1, 1, (2, 3)) = \alpha\delta$, and $(-1, -1, -1, -1, (2, 3)) = \alpha\gamma$.

Noticing that $\delta = \beta\gamma$, we have the symmetry group

$$G = \langle 1, \alpha, \beta, \gamma \rangle .$$

Since every element but 1 in G has order 2, then we get that G is Abelian. Therefore G is an Abelian group with order 8. By the Fundamental Theorem of Finite Generated Abelian Group, the structure of G is $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. But since every element except 1 in G has order 2, the structure of G is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus

$$G = \{1, \alpha\} \times \{1, \beta\} \times \{1, \gamma\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $\alpha = (1, 1, -1, -1, (2, 3))$, $\beta = (1, -1, -1, -1, e)$, and $\gamma = (-1, -1, 1, 1, e)$.

So the structure of the symmetry group of composite link $2_1^2 m \sharp 2_1^2$ is the same with the composite link $2_1^2 \sharp 2_1^2$, but has different elements in its symmetry group.

References

[1] Jason Catarella, *Notes on link tabulations*

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