

HT AREA AND PURE COMPLEX CROFTON MEASURE FOR COMPLEX L^1 SPACE

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ABSTRACT. We find a discrete Crofton measure on the space of affine lines $\overline{Gr_1(\mathbb{R}^2)}$ in real L^1 plane, and show that there is no pure complex Crofton measure for complex L^1 space.

1. DISCRETE CROFTON MEASURE FOR REAL L^1 PLANE

Let $(\mathbb{R}^2, \|\cdot\|_1)$ be the L^1 -plane. For any $P_1 := (x_1, y_1)$ and $P_2 := (x_2, y_2)$ in (\mathbb{R}^2, F) , we know, obviously, that

$$\|\overline{P_1 P_2}\|_1 = \|\overline{P_1 Q}\|_1 + \|\overline{QP_2}\|_1, \quad (1.1)$$

where $Q := (x_2, y_1)$. Furthermore, we have the following lemma.

Lemma 1.1. *Suppose $P_1 := (x_1, y_1)$ and $P_2 := (x_2, y_2)$ are distinct two arbitrary points in $(\mathbb{R}^2, \|\cdot\|_1)$, then all rectifiable “monotonic” curves joining P_1 and P_2 have the same L^1 -length, here a curve $C(t) : [a, b] \rightarrow \mathbb{R}^2$, $C(t) = (x(t), y(t))$ is called “monotone” if both $x(t)$ and $y(t)$ are monotonic functions.*

Proof. Since the length of a rectifiable curve is the least upper bound of the length of inscribed polygonal path and (1.1), then the lemma follows. \square

First, let us explore some properties of this space. In Figure 1.1 on page 2, the curve C is a rectifiable “monotonic” curve, so it has the same L^1 -length with the straight line segment $\overline{P_1 P_2}$. However, one always has

$$\#(l \cap \overline{P_1 P_2}) \leq \#(l \cap C) \quad (1.2)$$

for all $l \in \overline{Gr_1(\mathbb{R}^2)}$ but l coinciding $\overline{P_1 P_2}$.

Additionally, if one rotates $\overline{P_1 P_2}$ by angle α , denoted as $R_\alpha(\overline{P_1 P_2})$, then

$$0 < \frac{\sqrt{2}}{2} \|\overline{P_1 P_2}\|_1 \leq \|R_\alpha(\overline{P_1 P_2})\|_1 \leq \sqrt{2} \|\overline{P_1 P_2}\|_1. \quad (1.3)$$

Now, let us prove the following theorem.

Theorem 1.2. *The Crofton measure is supported on the set of lines which are horizontal or vertical in L^1 plane.*

Proof. Assume that there is a Crofton measure μ for the L^1 length, in other words,

$$\|\overline{P_1 P_2}\|_1 = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(l \cap \overline{P_1 P_2}) d\mu(l), \quad (1.4)$$

then $\#(l \cap \overline{P_1 P_2}) \leq \#(l \cap C)$ a.e. $l \in \overline{Gr_1(\mathbb{R}^2)}$.

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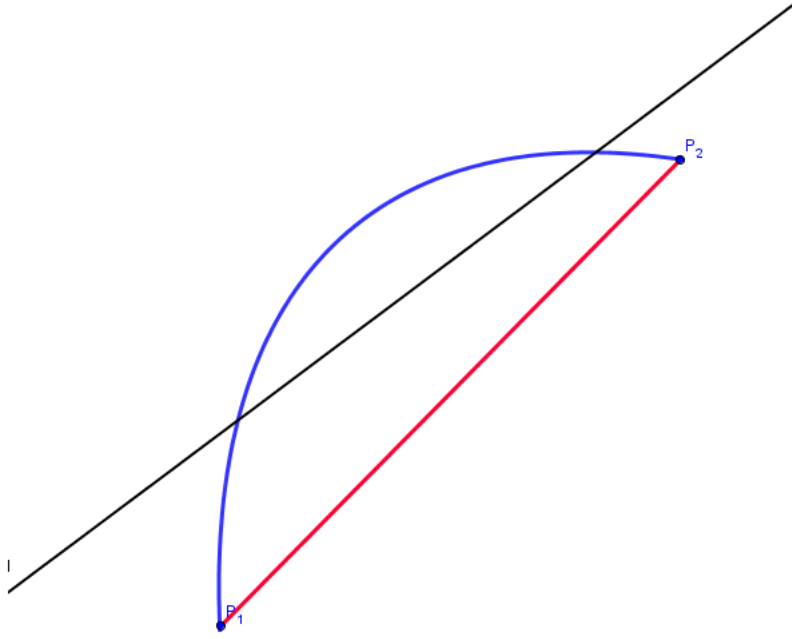


FIGURE 1.1. Paths Intersecting with Affine Lines

Choose P_1 to be the origin and $P_2 = (0, 1)$ on the y -axis, let $I := \overline{P_1 P_2}$, and define

$$\begin{aligned} \varphi : \overline{Gr_1(\mathbb{R}^2)} &\rightarrow S^1 \times \mathbb{R} \\ \varphi(l) &= (\theta, r) \end{aligned} \quad (1.5)$$

in which θ is the angle of l to x -axis and r is the euclidean distance of l to the origin. Then by (1.4),

$$(\varphi_*(\mu))([0, \pi] \times [0, 1]) = \mu(\{l \in \overline{Gr_1(\mathbb{R}^2)} : \#(l \cap I) > 0\}) = 1. \quad (1.6)$$

Moreover, if one chooses P_1 to be the origin, $P_2 = (r \cos \phi, r \sin \phi)$ and $Q = (0, r \sin \phi)$, see Figure 1.2 on page 3, then

$$\begin{aligned} &\varphi(\{l \in \overline{Gr_1(\mathbb{R}^2)} : \#(l \cap (\overline{P_1 Q} \cup \overline{Q P_2})) = 2\}) \\ &= \left\{ \left(\frac{\pi}{2} - \phi \right) \left(1 - \frac{h}{r \sin \phi} \right) \leq \theta \leq \frac{\pi}{2} : h \in [0, r \sin \phi] \right\} \end{aligned} \quad (1.7)$$

for $\phi \in (0, \frac{\pi}{2})$ and

$$\begin{aligned} &\varphi(\{l \in \overline{Gr_1(\mathbb{R}^2)} : \#(l \cap (\overline{P_1 Q} \cup \overline{Q P_2})) = 2\}) \\ &= \left\{ \frac{\pi}{2} \leq \theta \leq \phi + \frac{\pi - \phi}{r \sin \phi} h : h \in [0, r \sin \phi] \right\} \end{aligned} \quad (1.8)$$

for $\phi \in (\frac{\pi}{2}, \pi)$.

On the other hand, since $\overline{P_1 Q} \cup \overline{Q P_2}$ has the same length with $\overline{P_1 P_2}$, then, by (1.2) and (1.4), we have

$$\mu(\{l \in \overline{Gr_1(\mathbb{R}^2)} : \#(l \cap (\overline{P_1 Q} \cup \overline{Q P_2})) = 2\}) = 0. \quad (1.9)$$

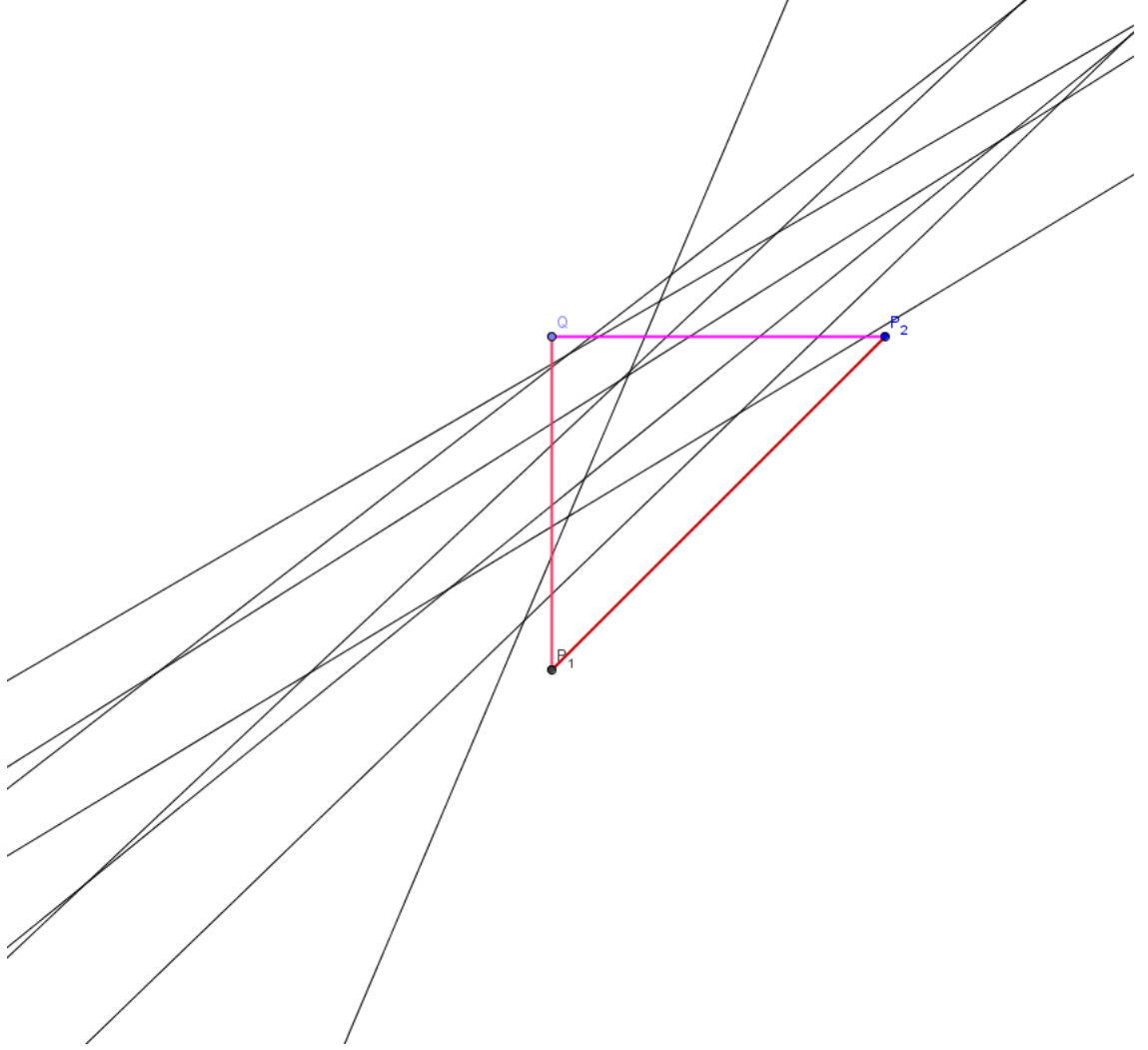


FIGURE 1.2. Lines with Two Intersection Points

Therefore, by (1.7) and (1.8),

$$(\varphi_*(\mu))\left(\left\{\left(\frac{\pi}{2} - \phi\right)\left(1 - \frac{h}{r \sin \phi}\right) \leq \theta \leq \frac{\pi}{2} : h \in [0, r \sin \phi]\right\}\right) = 0 \quad (1.10)$$

for $\phi \in (0, \frac{\pi}{2})$ and

$$(\varphi_*(\mu))\left(\left\{\frac{\pi}{2} \leq \theta \leq \phi + \frac{\pi - \phi}{r \sin \phi} h : h \in [0, r \sin \phi]\right\}\right) = 0 \quad (1.11)$$

for $\phi \in (\frac{\pi}{2}, \pi)$. By taking $r = \frac{1}{\phi}$ and $\phi \rightarrow 0$ in (1.10), and $r = \frac{1}{\pi - \phi}$ and $\phi \rightarrow \pi$ in (1.11), one obtains that

$$(\varphi_*(\mu))\left(\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)\right) \times [0, 1] = 0. \quad (1.12)$$

By rescaling $\overline{P_1 P_2}$, we conclude that

$$(\varphi_*(\mu))((0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)) \times \mathbb{R} = 0, \quad (1.13)$$

thus $\varphi_*(\mu)$ is supported on $\{0\} \times \mathbb{R}$ and $\{\frac{\pi}{2}\} \times \mathbb{R}$, which implies that μ is supported on the set of lines which are horizontal or vertical in L^1 plane. \square

Let's define a discrete measure supported on x -axis and y -axis by

$$(\delta_x + \delta_y)(l) := \begin{cases} 1 & \text{if } l \text{ coincides } x\text{-axis or } y\text{-axis,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.14)$$

We have the following theorem

Theorem 1.3. *The Crofton measure for real L^1 plane is $\delta_x + \delta_y$.*

Proof. For any straight line segment $P_1 P_1$ in L^1 plane, for which one can see Figure 1.1 on page 2, Since

$$\text{Length}(P_1 P_1) = |\pi_y(P_1 P_1)| + |\pi_x(P_1 P_1)|, \quad (1.15)$$

then

$$\text{Length}(P_1 P_1) = \int_{Gr_1(\mathbb{R}^2)} \#(l \cap P_1 P_1) d(\delta_x + \delta_y), \quad (1.16)$$

and so the claim follows. \square

2. CROFTON MEASURE ON $Gr_2(\mathbb{R}^4)$

Suppose E is contained in a complex line L in \mathbb{C}^2 with complex L^1 norm $\|(z, w)\| = |z| + |w|$, and μ is the Crofton measure, for which one can see [BF], for HT^2 , i.e.

$$HT^2(E) = \int_{Gr_2(\mathbb{R}^4)} \text{vol}_2(\pi_P(E)) d\mu. \quad (2.1)$$

From the intersection map

$$\pi : Gr_3(\mathbb{R}^4) \times Gr_3(\mathbb{R}^4) \setminus \Delta \rightarrow Gr_2(\mathbb{R}^4), \quad (2.2)$$

as we know, μ is supported on $\{\mathbb{C} \times \{0\}\} \cup \{\{0\} \times \mathbb{C}\} \cup T_{AB}$, where

$$T_{AB} := \{\text{span}(v_A, v_B) : v_A \in \mathbb{C} \times \{0\}, v_B \in \{0\} \times \mathbb{C}\}. \quad (2.3)$$

Since $T_{AB} \cong \mathbb{T}^2$, where $\mathbb{T}^2 := S_A \times S_B$ in which S_A and S_B are the unit circles in $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$ respectively, then in terms of the push-forward measure by (2.2), one has that

$$\int_{T_{AB}} \text{vol}(\pi_P(E)) d\mu = \int_{(\theta_A, \theta_B) \in \mathbb{T}^2} \text{vol}_2(\pi_{\text{span}(\theta_A, \theta_B)}(E)) d\theta_A d\theta_B. \quad (2.4)$$

However, $\text{vol}(\pi_{\text{span}(\theta_A, \theta_B)}(E))$ doesn't vanish for most $(\theta_A, \theta_B) \in \mathbb{T}^2$, so the integrals in (2.4) doesn't vanish. To see an example, we choose

$$L = \{(z, w) \in \mathbb{C}^2 : z = w\} = \{(x, y, x, y) \in \mathbb{R}^4\} \quad (2.5)$$

and E to be a real 2-disk in L . For $v_A = (x_1, x_2, 0, 0)$ and $v_B = (0, 0, v_3, v_4)$, let $v_A^\perp := (-x_2, x_1, 0, 0)$ and $v_B^\perp := (0, 0, -x_4, x_3)$, then

$$\mathbb{R}^4 = \text{span}(v_A, v_B) \oplus \text{span}(v_A^\perp, v_B^\perp). \quad (2.6)$$

Therefore

$$(x, y, x, y) = \left(\frac{xx_1 + yy_2}{x_1^2 + x_2^2} v_A + \frac{xx_3 + yy_4}{x_3^2 + x_4^2} v_B \right) + \left(\frac{yx_1 - xx_2}{x_1^2 + x_2^2} v_A^\perp + \frac{yx_3 - xx_4}{x_3^2 + x_4^2} v_B^\perp \right) \quad (2.7)$$

and so

$$\pi_{\text{span}(v_A, v_B)}((x, y, x, y)) = \frac{xx_1 + yy_2}{x_1^2 + x_2^2} v_A + \frac{xx_3 + yy_4}{x_3^2 + x_4^2} v_B \quad (2.8)$$

for any $(x, y, x, y) \in L$. Thus $\pi_{\text{span}(v_A, v_B)}(L) = \text{span}(v_A, v_B)$ as long as

$$\det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \neq 0. \quad (2.9)$$

Therefore it is easy to see that $\text{vol}_2(\pi_{\text{span}(\theta_A, \theta_B)}(E)) \neq 0$ provided (2.9) that holds for most $\text{span}(v_A, v_B) \in T_{AB}$.

3. CROFTON MEASURE ON $Gr_1(\mathbb{C}^2)$

Let $L_\theta := \text{span}((\cos \theta, \sin \theta)_\mathbb{C})$ be a complex line in \mathbb{C}^2 , and U be the rectangle spanned by $(\cos \theta, \sin \theta)$ and $(\sqrt{-1} \cos \theta, \sqrt{-1} \sin \theta)$, then

$$\begin{aligned} \text{area}(\pi_{\text{span}(\theta_A, \theta_B)}(U)) &= \left| \det \begin{pmatrix} \cos \theta \cos \theta_A & \sin \theta \cos \theta_B \\ \cos \theta \sin \theta_A & \sin \theta \sin \theta_B \end{pmatrix} \right| \\ &= |\sin \theta \cos \theta \sin(\theta_B - \theta_A)|, \end{aligned} \quad (3.1)$$

therefore

$$\int_{(\theta_A, \theta_B) \in \mathbb{T}^2} \text{area}(\pi_{\text{span}(\theta_A, \theta_B)}(U)) d\theta_A d\theta_B = |\sin \theta \cos \theta|. \quad (3.2)$$

On the other hand, for any complex line $L_{\phi, \psi} := \text{span}((\cos \phi, e^{\sqrt{-1}\psi} \sin \phi)_\mathbb{C})$ in \mathbb{C}^2 ,

$$\begin{aligned} &\text{area}(\pi_{L_{\phi, \psi}}(U)) \\ &= \left| \det \begin{pmatrix} \cos \theta \cos \phi + \sin \theta \sin \phi \cos \psi & -\sin \theta \sin \phi \sin \psi \\ \sin \theta \sin \phi \sin \psi & \cos \theta \cos \phi + \sin \theta \sin \phi \cos \psi \end{pmatrix} \right| \\ &= \cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + 2 \sin \theta \cos \theta \sin \phi \cos \phi \cos \psi. \end{aligned} \quad (3.3)$$

As we know, the problem on the existence of Crofton measure on complex lines for area becomes whether there exists some function $f(\phi)$, such that

$$\int_0^{2\pi} \int_0^{2\pi} \text{area}(\pi_{L_{\phi, \psi}}(U)) f(\phi) d\phi d\psi = \int_{(\theta_A, \theta_B) \in \mathbb{T}^2} \text{area}(\pi_{\text{span}(\theta_A, \theta_B)}(U)) d\theta_A d\theta_B. \quad (3.4)$$

But from (3.3),

$$\int_0^{2\pi} \int_0^{2\pi} \text{area}(\pi_{L_{\phi, \psi}}(U)) f(\phi) d\phi d\psi = \int_0^{2\pi} (\cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi) f(\phi) d\phi, \quad (3.5)$$

thus by (3.2) we need to have

$$\int_0^{2\pi} (\cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi) f(\phi) d\phi = |\sin \theta \cos \theta|. \quad (3.6)$$

Taking $\theta = 0$ and $\frac{\pi}{2}$ in (3.6), we have

$$\int_0^{2\pi} \cos^2 \phi f(\phi) d\phi = \int_0^{2\pi} \sin^2 \phi f(\phi) d\phi = 0, \quad (3.7)$$

which implies $\int_0^{2\pi} f(\phi) d\phi = 0$.

However, let $\theta \in (0, \frac{\pi}{2})$ and take the derivative with respect to θ in (3.6), then one has

$$\int_0^{2\pi} (-\sin 2\theta \cos 2\phi) f(\phi) d\phi = \cos 2\theta, \quad (3.8)$$

which implies that $\int_0^{2\pi} f(\phi) \cos 2\phi d\phi = -\cot 2\theta$, that is a contradiction.

So we obtain the following

Theorem 3.1. *There is no Crofton measure which is only supported on complex lines for complex L^1 space.*

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REFERENCES

- [BF] A. Bernig, J. H. G. Fu, Convolution of convex valuations, *Geometriae Dedicata*, 2006, Springer