ON HOLMES-THOMPSON AREA: HT VALUATION THEORY UNDER "MICROSCOPE"

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1. Length and Related

The classic Crofton formula is

$$Length(\gamma) = \frac{1}{4}\int_0^\infty \int_0^{2\pi} \#(\gamma \cap l(r,\theta))d\theta dr \tag{1.1}$$

for any rectifiable curve in Euclidean plane, where θ is the angle of the normal of the oriented line l to the x-axis and r is its distance to the origin. Let us denote the affine 1-Grassmannians (lines) in $\mathbb{R}^2$ by $\overline{Gr_1(\mathbb{R}^2)}$.

As for Minkowski plane, it is a normed two dimensional space with a norm $F(\cdot) = ||\cdot||$, in which the unit disk is convex and F has some smoothness.

Two of the key tools used to obtain the Crofton formula for Minkowski plane are the cosine transform and Gelfand transform. Let us explain them one by one first and see their connection next. A fact from spherical harmonics about cosine transform is there is some even function on S^1 such that

$$F(\cdot) = \frac{1}{4}\int_{S^1} |\langle \xi, \cdot \rangle| g(\xi) d\xi, \tag{1.2}$$

if F is an even C^4 function on S^1. A good reference for this is [G]. As for Gelfand transform, it is the transform of differential forms and densities on double fibrations, for instance, $\mathbb{R}^2 \overset{\pi_1}{\leftarrow} \mathcal{I} \overset{\pi_2}{\rightarrow} \overline{Gr_1(\mathbf{\mathbb{R}^2})}$, where $\mathcal{I} := \left\{(x,l) \in \mathbb{R}^2 \times \overline{Gr_1(\mathbf{\mathbb{R}^2})} : x \in l\right\}$ is the incidence relations and π_1 and π_2 are projections. A formula one can take as an example of the fundamental theorem of Gelfand transform is the following

$$\int_\gamma \pi_{1*}\pi_2^*|\Omega| = \int_{l\in\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l)|\Omega|, \tag{1.3}$$

where $\Omega := g(\theta)d\theta \wedge dr$. But we give a direct proof here.

Proof. First, consider the case of $\Omega = d\theta \wedge dr$. For any $v \in T_x\gamma$, since there is some $v' \in T_{x'}\mathcal{I}$, such that $\pi_{1*}(v') = v$, then

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$$\begin{aligned}(\pi_{1*}\pi_2^*|\Omega|)_x(v) &= (\int_{\pi_1^{-1}(x)} \pi_2^*|\Omega|)_x(v)\\ &= \int_{x'\in\pi_1^{-1}(x)}(\pi_2^*|\Omega|)_{x'}(v')\\ &= \int_{S^1}(\pi_2^*|d\theta\wedge dr|)(v')\\ &= \int_{S^1}|dr(\pi_{2*}(v'))|d\theta\\ &= \int_{S^1}|\langle v,\theta\rangle|d\theta\\ &= 4|v|.\end{aligned} \tag{1.4}$$

So $\int_\gamma \pi_{1*}\pi_2^*|\Omega| = 4Length(\gamma) = \int_{l\in\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma\cap l)|\Omega|$ by the classic Crofton formula.

When $\Omega = f(\theta)d\theta\wedge dr$, we just need to replace $d\theta$ by $g(\theta)d\theta$ in the equalities in the first case. □

Moreover, from the above proof and (1.2), for any curve $\gamma(t) : [a,b] \to \mathbb{R}^2$ differentiable almost everywhere in the Minkowski space,

$$\int_\gamma \pi_{1*}\pi_2^*|\Omega| = \int_a^b (\pi_{1*}\pi_2^*|\Omega|)(\gamma'(t))dt = \int_a^b 4F(\gamma'(t))dt = 4Length(\gamma), \tag{1.5}$$

so then by (1.3) we know

$$Length(\gamma) = \frac{1}{4}\int_{l\in\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma\cap l)|g(\theta)d\theta\wedge dr| \tag{1.6}$$

for Minkowski plane.

The Holmes-Thompson Area $HT^2(U)$ of a measurable set U in a Minkowski plane is defined as $HT^2(U) := \frac{1}{\pi}\int_{D^*U}|\omega_0|^2$, where ω_0 is the natural symplectic form on the cotangent bundle of $\mathbb{R}^2$ and $D^*U := \{(x,\xi)\in T^*\mathbb{R}^2 : F^*(\xi)\le 1\}$. To study it from the perspective of integral geometry, we need to introduce a symplectic form ω on the space of affine lines $\overline{Gr_1(\mathbb{R}^2)}$, that one can see [A].

2. HT Area and Related

Now let's see the Crofton formula for Minkowski plane, which is $Length(\gamma) = \frac{1}{4}\int_{\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma\cap l)|\omega|$. To prove this, it is sufficient to show that it holds for for any straight line segment

$$L : [0, ||p_2 - p_2||] \to \mathbb{R}^2,\ L(t) = p_1 + \frac{p_2 - p_1}{||p_2 - p_1||}t, \tag{2.1}$$

starting at p_1 and ending at p_2 in $\mathbb{R}^2$. First, using the diffeomorphism between the circle bundle and co-circle bundle, which is

$$\begin{aligned}\varphi_F &: S\mathbb{R}^2 \to S^*\mathbb{R}^2\\ \varphi_F(x,\xi) &= (x, dF_\xi),\end{aligned} \tag{2.2}$$

we can obtain a fact that

$$\begin{aligned}\int_{L\times\{\frac{p_2-p_1}{||p_2-p_1||}\}}\varphi_F^*\alpha_0 &= \int_{\varphi_F(L\times\{\frac{p_2-p_1}{||p_2-p_1||}\})}\alpha_0\\ &= \int_0^{||p_2-p_1||}\alpha_{0dF_{\frac{p_2-p_1}{||p_2-p_1||}}}((\tfrac{p_2-p_1}{||p_2-p_1||},0))dt\\ &= \int_0^{||p_2-p_1||}dF_{\frac{p_2-p_1}{||p_2-p_1||}}(\tfrac{p_2-p_1}{||p_2-p_1||})dt,\end{aligned}\tag{2.3}$$

where α_0 is the tautological one-form, precisely $\alpha_{0\xi}(X):=\xi(\pi_{0*}X)$ for any $X\in T_\xi T^*\mathbb{R}^2$, and $d\alpha_0=\omega_0$. Applying the the basic equality that $dF_\xi(\xi)=1$, which is derived from the positive homogeneity of F, for all $\xi\in S\mathbb{R}^2$, the above quantity becomes $\int_0^{||p_2-p_1||}1dt$, which equals to $||p_2-p_1||$.

Let $R:=\{\xi_x\in S^*\mathbb{R}^2 : x\in\overline{p_1p_2}\}$ and $T=\left\{l\in\overline{Gr_1(\mathbb{R}^2)} : l\cap\overline{p_1p_2}\neq\emptyset\right\}$, and p' is the projection (composition) from $S^*\mathbb{R}^2$ to $\overline{Gr_1(\mathbb{R}^2)}$.

Apply the above fact and $p'^*\omega=\omega_0$,

$$\begin{aligned}\int_T|\omega| = \int_{p'(R)}|\omega| = \int_R|p'^*\omega| &= \int_R|\omega_0|\\ &= |\int_{R^+}\omega_0|+|\int_{R^-}\omega_0|\\ &= |\int_{\partial R^+}\alpha_0|+|\int_{\partial R^-}\alpha_0|\\ &= 4||p_2-p_1||.\end{aligned}\tag{2.4}$$

Thus we have shown the Crofton formula for Minkowski plane.

Furthermore, combining with (1.6), we have

$$\frac{1}{4}\int_{l\in\overline{Gr_1(\mathbb{R}^2)}}\#(\gamma\cap l)|\Omega| = \frac{1}{4}\int_{\overline{Gr_1(\mathbb{R}^2)}}\#(\gamma\cap l)|\omega|,\tag{2.5}$$

where $\Omega=g(\theta)d\theta\wedge dr$. Then, by the injectivity of cosine transform in [G], $|\Omega|=|\omega|$.

To obtain the HT area, one can define a map

$$\begin{aligned}\pi:\overline{Gr_1(\mathbb{R}^2)}\times\overline{Gr_1(\mathbb{R}^2)}\setminus\tilde{\triangle}\to\mathbb{R}^2\\ \pi(l,l')=l\cap l',\end{aligned}\tag{2.6}$$

where $\tilde{\triangle}:=\{(l,l') : l\parallel l' \text{ or } l=l'\}$, extended from Alvarez's construction of taking intersections, [AF]. The following theorem can be obtained.

Theorem 2.1. *$HT^2(U)=\frac{1}{2\pi}\int_{x\in\mathbb{R}^2}\chi(x\cap U)|\pi_*\Omega^2|$ for any bounded measurable subset U of a Minkowski plane.*

Proof. On one hand,

$$\frac{1}{\pi}\int_{D^*U}\omega_0^2 = \frac{1}{\pi}\int_{\partial D^*U}\omega_0^2 = \frac{1}{\pi}\int_{S^*U}\alpha_0\wedge\omega_0.\tag{2.7}$$

On the other hand, let $\mathcal{T}_U:=\left\{((l,l')\in\overline{Gr_1(\mathbb{R}^2)}\times\overline{Gr_1(\mathbb{R}^2)} : l\cap l'\in U\right\}$,

$$\frac{1}{\pi}\int_{x\in\mathbb{R}^2}\chi(x\cap U)\pi_*\Omega^2 = \frac{1}{\pi}\int_U\pi_*\omega^2 = \frac{1}{\pi}\int_{\mathcal{T}_U}\omega^2.\tag{2.8}$$

Let $\mathbb{T}^*U := \{(\xi_x, \xi'_x) : \xi_x, \xi'_x \in S^*_xU\}$, then

$$(p' \times p')^{-1}(\mathcal{T}_U) = \mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S^*_xU\}\,. \tag{2.9}$$

Therefore

$$\begin{aligned}\frac{1}{\pi}\int_{\mathcal{T}_U}\omega^2 &= \frac{1}{\pi}\int_{\mathbb{T}^*U\setminus\{(\xi_x,\xi_x):\xi_x\in S^*_xU\}} p'^*\omega^2\\ &= \frac{1}{\pi}\int_{\mathbb{T}^*U\setminus\{(\xi_x,\xi_x):\xi_x\in S^*_xU\}} \omega_0^2\\ &= \frac{2}{\pi}\int_{\{(\xi_x,\xi_x):\xi_x\in S^*_xU\}} \alpha_0\wedge\omega_0\\ &= \frac{2}{\pi}\int_{S^*U}\alpha_0\wedge\omega_0.\end{aligned} \tag{2.10}$$

So the claim follows from (2.7),(2.8) and (2.10). □

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