Topology of Manifolds by Sectional Curvature

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- Exponential map: $exp_p : T_pM \to M$, $exp_p(v) = \gamma_v(1)$. Critical point of the exponential map; Local diffeomorphism; and Local isometry for zero sectional curvature space.

Theorem

(Second Variational Formula) Let $\gamma : [0, a] \to M$ *be a geodesic,* $f : [0, a] \times (-\epsilon, \epsilon) \to M$ *be a variation of* γ*, and* $V(t) = \frac{\partial f}{\partial s}(t, 0)$*. Then, for the energy functional* $E(s)$*,*

$$\begin{aligned} E''(0) &= \int_0^a \langle V'(t), V'(t)\rangle_g - \langle R(V(t), \gamma'(t))\gamma'(t), V(t)\rangle_g dt \\ &\quad -\langle \frac{\partial}{\partial s}\frac{\partial f}{\partial t}(0,0), \gamma'(0)\rangle_g + \langle \frac{\partial}{\partial s}\frac{\partial f}{\partial s}(0,a), \gamma'(a)\rangle_g. \end{aligned}$$

Theorem

(Synge) Any compact connected oriented Riemannian manifold of even dimension with positive sectional curvature has trivial fundamental group.

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- Construct a variation $f(t,s) = exp_{\gamma(t)} sV(t)$, where $V(t)$ is the parallel transport of v along $\gamma(t)$, which satisfies $V' = 0$ and $\frac{\partial f}{\partial t}(0,0) = \frac{\partial f}{\partial s}(0,a)$.

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- $E''(0) < 0$, $E'(0) = 0 \Longrightarrow E(s) < E(0)$ for small s.

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$\square$

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e.g. $S^2 \setminus \{N, S\}$, $\mathbb{RP}^3$, $\mathbb{RP}^2$, etc.

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Remark

Non-orietable M has a double orientable covering.

Theorem

No Riemannian metric can make $\mathbb{RP}^n \times \mathbb{RP}^n$ have positive sectional curvature everywhere.

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Conjecture

(Hopf) Riemannian manifold with the same assumptions as in Synge theorem has positive Euler characteristic. No metric with positive sectional curvature on $S^2 \times S^2$.

Theorem

(Preissman) Any nonidentity abelian subgroup of the fundamental group of a compact manifold with negative sectional curvature is cyclic.

Theorem

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Remark

As we know, the flat n-torus $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ has fundamental group $\mathbb{Z}^n$; The quotient space of $\mathbb{T}^n$ acted by a finite group.


S. Nishikawa (Translated by K. Abe), Variational Problem in Geometry, Publishers, Tokyo, 1998


X. Rong, On the Fundamental Groups of Manifolds of Positive Sectional Curvature The Annals of Mathematics, Vol. 143, 1996


B.Wilkin, Manifolds with positive sectional curvature almost everywhere, Invent. math., 2002

Thank you for attending!