Chapter 1 Functions, Graphs, and Models

The principal objective of calculus is the analysis of problems of change (of motion, for example) and of content (the computation of area and volume, for instance). It provides a framework for modeling system in which there is change, and a way to deduce the prediction of such models.

Functions

The key to the mathematical analysis is the recognition of relationships. such a relationship may be a formula that express one variable as a function of another.

- The area A of a circle of radius r is given by $A = \pi r^2$. The volume V and surface area S of a sphere of radius r are given by
$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$
- After t second(s) a body that has been dropped from rest has fallen a distance
$$s = \frac{1}{2}gt^2$$
feet(ft) and has speed $v = gt$ feet per second (ft/s), where g $\approx$ 32ft/s^2 is gravitational acceleration.

Definition

A real-valued function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.

The set D of all numbers for which $f(x)$ is defined is called the domain of the function f. The number $f(x)$ is called the value of the function f at the number x. The set of all values $y = f(x)$ is called the range of f.

Example 1
$$f(x) = x^2$$

Example 2 $f(x) = x^2 + x - 3$

some typical values of f are $f(-2) = -1, f(0) = -3, f(3) = 9,$

$$f(4) = 4^2 + 4 - 3 = 17$$

$$f(c) = c^2 + c - 3$$

$$f(2+h) = (2+h)^2 + (2+h) - 3 = h^2 + 5h + 3$$

$$f(-t^2) = (-t^2)^2 + (-t^2) - 3 = t^4 - t^2 - 3.$$

when we describe the function f by writing a formula $y = f(x)$, we call x the independent variable and y the dependent variable.

$$x \xrightarrow{f} f(x)$$

Example 3 Not every function has a rule expressible as a simple one-part formula. For instance, if we write

$$h(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ \sqrt{-x} & \text{if } x < 0. \end{cases}$$

Then we have defined a perfectly good function with domain R. Some of its values are $h(-4) = 2, h(0) = 0, h(2) = 4$

Domains and Intervals

Graphs of equations and functions

Recall the slope-intercept equation of the straight line with slope $m = tan\phi$, angle of inclination ϕ, and y-intercept b :

$$y = mx + b$$

The "rise over run " definition

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

the point-slope equation of the straight line

$$y - y_0 = m(x - x_0)$$

Definition

The graph of an equation in two variables x and y is the set of all points (x, y) in the xy-plane that satisfy the equation.

Recall the graph of the equation

$$(x-h)^2 + (y-k)^2 = r^2$$

is the circle of radius r centered at the origin (h, k). This follows from the distance formula.

Translation Principle

When the graph of an equation is translated h units to the right and k units upward, the equation of the translated curve is obtained from the original equation by replacing x with x-h and y with y-k.

Definition

The graph of the function f is the graph of the equation $y = f(x)$.

Example 4 the graph of function $f(x) = |x|$.

Example 5 sketch of the graph of the reciprocal function

$$f(x) = \frac{1}{x}$$

Parabolas

The graph of a quadratic function of the form

$$f(x) = ax^2 + bx + c (a \neq 0)$$

is a *parabola.*

(graph? vertex? symmetric axis)

Power functions

A function of the form $f(x) = x^k$ (where k is a constant) is called a power function. If $k = 0$ then we have the constant function $f(x) \equiv 1$.

The shape of the graph of a power function with exponent $k = n$, a positive integer, depends on whether n is even or odd.

Combinations of functions

Suppose f and g are functions and that c is a fixed real number. The (scalar) multiple cf, the sum $f+g$, the difference $f-g$, the product $f \cdot g$ and the quotient f/g are the new functions with the following formulas:

$$(cf)(x) = c \cdot f(x),$$
$$(f+g)(x) = f(x) + g(x),$$

Definition

the composition of the two functions f and g is the function $h = f \circ g$ defined by

$$h(x) = f(g(x))$$

for all x in the domain of g such that $u = g(x)$ is in the domain of f.

Example 6 let $f(x) = x^2 + 1$ and $g(x) = x - 1$, then

$$(3f)(x) = 3(x^2 + 1)$$
$$(f+g)(x) = (x^2+1) + (x-1) = x^2 + x$$
$$(f-g)(x) = (x^2+1) - (x-1)$$
$$(f \cdot g)(x) = f(g(x)) = f(x-1) = (x-1)^2 + 1$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2+1}{x-1}$$

Polynomials

A polynomial of degree n is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the coefficients $a_0, a_1, \cdots, a_n$ are fixed real numbers and $a_n \neq 0$.

The first-degree polynomial is simply a *linear function* $a_1 x + a_0$ whose graph is a straight line. A second-degree polynomial is a *quadratic function* $y = a_2 x^2 + a_1 x + a_0$ whose graph is a parabola.

Rational functions

A rational function is a quotient

$$f(x) = \frac{p(x)}{q(x)}$$

Algebraic functions

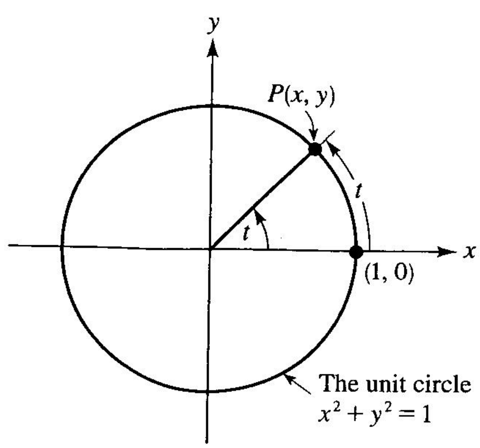
An algebraic function is one whose formula can be constructed beginning with power functions and applying the algebraic operations of additions, subtraction, multiplication by a real number, multiplication, division, and/or root-taking.

Transcendental functions
The trigonometric, exponential, and logarithmic functions are called *transcendental* functions.

Trigonometric functions

first defined in a right triangle, then in a unit circle, then for all real numbers.

Definitions: Trigonometric Functions of Real Numbers
Let $P(x, y)$ denote the point on the unit circle that has arc length t from $(1, 0)$. Or equivalently, let $P(x, y)$ denote the point where the terminal side of the angle with radian measure t intersects the unit circle. (Note: $t > 0$ if the rotation is counter clockwise and $t < 0$ if the rotation is clockwise) Then the six trigonometric functions of the real number t are defined as follows.



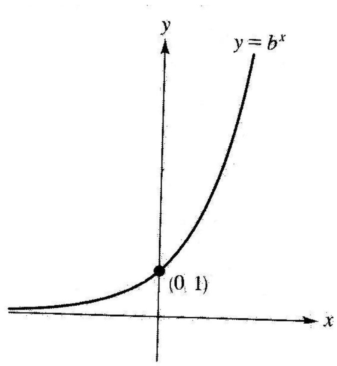
$$sin(t) = y \quad csc(t) = 1/y$$
$$cos(t) = x \quad sec(t) = 1/x$$
$$tan(t) = y/x \quad cot(t) = x/y$$

Exponential functions and logarithmic functions

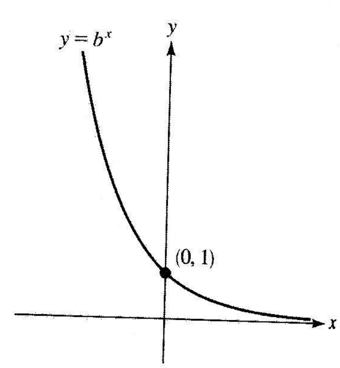
Definition: Let $b > 0$ and n a positive integer, then
(1) $b^0 = 1$, for instance: $2^0 = 1$
(2) $b^n = \underbrace{b \cdot b \cdots b}_{nfactors}$, for instance: $2^3 = 2 \cdot 2 \cdot 2 = 8$
(3) $b^{-n} = \frac{1}{b^n}$, for instance: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
(4) $b^{\frac{1}{n}} = \sqrt[n]{b}$,(note that: $(b^{\frac{1}{n}})^n = (\sqrt[n]{b})^n = b$) for instance: $2^{\frac{1}{3}} = \sqrt[3]{2}$
(5) $b^{\frac{m}{n}} = (b^{\frac{1}{n}})^m = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$, for instance: $2^{\frac{2}{5}} = \sqrt[5]{2^2}$

Definition: The Exponential Function with Base b:
Let $b > 0$ and $b \neq 1$, then f(x) is an exponential function with base b if $f(x) = b^x$.
Domain:$(-\infty, \infty)$, **Range:** $(0, \infty)$;
Horizontal Asymptote: x-axis;
y-intercept:1;
x-intercept: None.

If $b > 1$, $f(x)$ is an increasing function, that is, $x_1 < x_2$ iff $b^{x_1} < b^{x_2}$.
If $0 < b < 1$, $f(x)$ is a decreasing function, that is, $x_1 < x_2$ iff $b^{x_1} > b^{x_2}$.



$b > 1$



$0 < b < 1$

Laws of Exponents:
(1) $b^{x_1} \cdot b^{x_2} = b^{x_1+x_2}$, for instance: $2^2 \cdot 2^3 = 2^5$
(2) $b^{x_1}/b^{x_2} = b^{x_1-x_2}$, for instance: $2^5/2^3 = 2^2$
(3) $(b^{x_1})^{x_2} = b^{x_1 x_2}$, for instance: $(2^2)^3 = 2^{2\cdot 3} = 2^6$

Definition: e is a constant and $e = 2.718......$

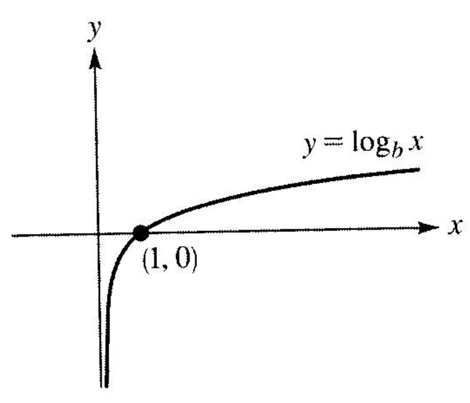
Definition: Let $b > 0$ and $b \neq 1$, $a > 0$, $log_b a$ is the exponent to which b must be raised to get a, that is,

$$c = log_b a \text{ iff } b^c = a,$$

Note $log_b b^a = a$ and $b^{log_b a} = a$. For instance, $log_2 2^3 = 3$ and $2^{log_2 8} = 8$.

Definition: Let $b > 0$ and $b \neq 0$, $a > 0$, the logarithmic function is one of the form $f(x) = log_b x$.

Domain: $(0, \infty)$, Range: $(-\infty, \infty)$
vertical Asymptote: y-axis, x-intercept: 1
If $b > 1$, $f(x)$ is an increasing function, that is, $x_1 < x_2$ iff $log_b x_1 < log_b x_2$.
If $0 < b < 1$, $f(x)$ is a decreasing function, that is, $x_1 < x_2$ iff



$log_b x_1 > log_b x_2$.

Note: Natural logarithm: $lnx = log_e x$, and $logx = log_{10} x$.

Remark: $c = log_b a$ is the **logarithmic form** of the **exponential form** of $b^c = a$.

Algebraic properties of log **function:**

If $x_1 > 0$ and $x_2 > 0$, then

(1) $log_b x_1 \cdot x_2 = log_b x_1 + log_b x_2$, for instance: $log_2(16 \cdot 4) = log_2 16 + log_2 4$.

(2) $log_b \frac{x_1}{x_2} = log_b x_1 - log_b x_2$, for instance: $log_2(\frac{16}{4}) = log_2 16 - log_2 4$.

(3) $log_b x_1^{x_2} = x_2 log_b x_1$, for instance: $log_2(8^2) = 2 log_2 8$.