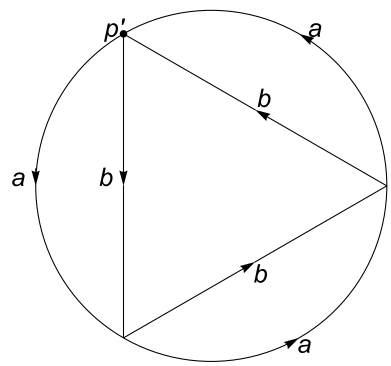

FIGURE 1. Figure for solution to problem 3

Topology Qualifying Exam Fall 2023 - Solutions

1. Let X be a compact space and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of nonempty closed subsets of X such that $A_{n+1} \subset A_n$ for all n. Prove that $\cap_{n=1}^{\infty} A_n$ is nonempty.

Solution: Let $U_n = X \setminus A_n$. Then each U_n is an open set since A_n is closed, and $U_n \subset U_{n+1}$ since $A_{n+1} \subset A_n$. Suppose for contradiction that $\cap_{n=1}^{\infty} A_n$ was empty. This would yield

$$\varnothing = \cap_{n=1}^{\infty}(X \setminus U_n) = X \setminus \cup_{n=1}^{\infty} U_n,$$

and hence $\cup_{n=1}^{\infty} U_n = X$. Since X is compact and the U_n are open, it would follow that finitely many U_n suffice to cover X, say $X = \cup_{n=1}^{N} U_n$. But since $U_n \subset U_{n+1}$ we have $\cup_{n=1}^{N} U_n = U_N$, so our contradiction assumption would imply that $X = U_N$ for some N, and hence that $A_N = X \setminus U_N$ is empty, contrary to the hypothesis that all of the A_n are nonempty.

2. Give an example of a continuous surjective function that is not an open map.

Solution: Let $X = [0,1] \cup [2,3]$ and $Y = [0,1]$, both with the subspace topology from the standard topology on $\mathbb{R}$. Define f by $f(x) = x$ when $x \in [0,1]$ and $f(x) = 1/2$ when $x \in [2,3]$. This is clearly surjective, and is continuous by the pasting lemma. But $[2,3]$ is open in Y while $f([2,3]) = \{1/2\}$ which is not open in $[0,1]$.

3. Let G be the free group on two generators a and b. Use covering space theory to find (giving an explicit description of the generators in terms of a and b) a normal, index-three subgroup of G.

Solution: Consider the covering space X' of $S^1 \vee S^1$ depicted in Figure 1, with the covering map $\pi : X' \to S^1 \vee S^1$ sending edges marked a to one of the circles and edges marked b to the other circle with orientations as indicated, and with the points at which 'a' and 'b' edges intersect (including the basepoint p' of X') mapping to the point of intersection p of the two circles. This is a three-sheeted covering space, which is normal because successive rotations of the figure by $\frac{2\pi}{3}$ map any point in the preimage of p to any other such point. Hence covering space theory implies that $\pi_* (\pi_1(X', p'))$ will be an index-three, normal subgroup of $\pi_1(S^1 \vee S^1, p)$, the latter of which is the free group on two generators, one for each of the two circles.

The space X' deformation retracts to a wedge of four circles (by contracting a maximal tree, which will consist of two edges, to a point), so its fundamental group is free on four generators; to see where these generators are mapped under π_* we concatenate edge labels for appropriate loops based at p' in the diagram (with inverses when we go against the orientation of a path). There are various possible choices of sets of generators; one such is $\{ab^{-1}, b^3, bab, a^{-1}b\}$, as the associated loops in the diagram map to the four distinct circles in $S^1 \vee S^1 \vee S^1 \vee S^1$ under the quotient that collapses the two edges of the triangle with vertex p'. Thus the desired index three normal subgroup of the free group on two generators is the one generated by ab^{-1}, b^3, bab, and $a^{-1}b$.

4. Let D denote the closed unit disk in $\mathbb{R}^2$ with boundary S^1. Choose $p \in S^1$ and let X denote the union $(S^1 \times S^1) \cup (D \times \{p\})$. Find the fundamental group and the homology of X.

Solution: X can be formed as a cell complex, consisting of the usual cell complex structure for the torus $S^1 \times S^1$ (with one zero cell p, two one-cells a and b, and a single two-cell e glued via the word $aba^{-1}b^{-1}$) together with an additional two-cell f glued via a. The fundamental group therefore has presentation $\langle a, b | aba^{-1}b^{-1}, a\rangle$ which simplifies to $\langle b\rangle$, *i.e.* the fundamental group is isomorphic to $\mathbb{Z}$.

For the homology we calculate the cellular chain complex (C_*, ∂). (Write ∂_k for the restriction of ∂ to C_k.) C_0 is generated by p, with $\partial_0 p = 0$. C_1 is generated by a and b, with $\partial_1 a = \partial_1 b = p - p = 0$. C_2 is generated by e and f, with $\partial_2 e = a + b - a - b = 0$ and $\partial_2 f = a$. All other C_k are 0. Hence

$$H_0(X) = \frac{\ker \partial_0}{\mathrm{Img}\partial_1} = \langle p\rangle \cong \mathbb{Z}, \quad H_1(X) = \frac{\ker \partial_1}{\mathrm{Img}\partial_2} = \frac{\langle a, b\rangle}{\langle a\rangle} \cong \mathbb{Z}, \quad H_2(X) = \frac{\ker \partial_2}{\mathrm{Img}\partial_3} = \langle e\rangle \cong \mathbb{Z},$$

and all other $H_k(X)$ are 0.

5. Find, with proof, a choice of identifications in pairs between the edges of a regular octagon such that the quotient space is homeomorphic to the Klein bottle.
Solution: There are multiple correct solutions to this problem. Any choice of identifications in pairs of the sides of an octagon yields a cell complex with four 1-cells and one 2-cell, while the number (say c_0) of 0-cells will depend on how the edges are identified. The Euler characteristic will thus be $c_0 - 4 + 1 = c_0 - 3$, so since the Euler characteristic of the Klein bottle is zero we will need to have $c_0 = 3$. Up to homeomorphism, there are precisely two compact connected surfaces without boundary having Euler characteristic zero, namely the Klein bottle K and the torus T. One way of distinguishing these is by the fact that $H_2(K) = 0$ while $H_2(T) = \mathbb{Z}$; in terms of the cellular chain complex this translates to the statement that the boundary of the 2-cell will be nonzero for K and zero for T. So an edge identification will work as long is it results in exactly three equivalence classes of vertices, and the word in the one-cells formed by circulating counterclockwise around the octagon does not have all generators appear in canceling pairs.

One way of achieving this is shown in Figure 2. Another valid way of arriving at that particular figure is by noting that $K \cong \mathbb{R}P^2 \# \mathbb{R}P^2$ and then gluing two presentations of $\mathbb{R}P^2$ as the quotient of a square by the restriction to its boundary of a 180° rotation.

6. Prove that $\mathbb{R}^3$ is not homeomorphic to $\mathbb{R}^4$.

Solution: Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^4$ is a homeomporphism. Let $p = (0, 0, 0) \in \mathbb{R}^3$ and let $q = f(p) \in \mathbb{R}^4$. Then f restricts to a homeomporphism from $\mathbb{R}^3 \setminus \{p\}$ to $\mathbb{R}^4 \setminus \{q\}$. But $\mathbb{R}^3 \setminus \{p\}$ deformation retracts on S^2 and $\mathbb{R}^4 \setminus \{q\}$ is homeomorphic

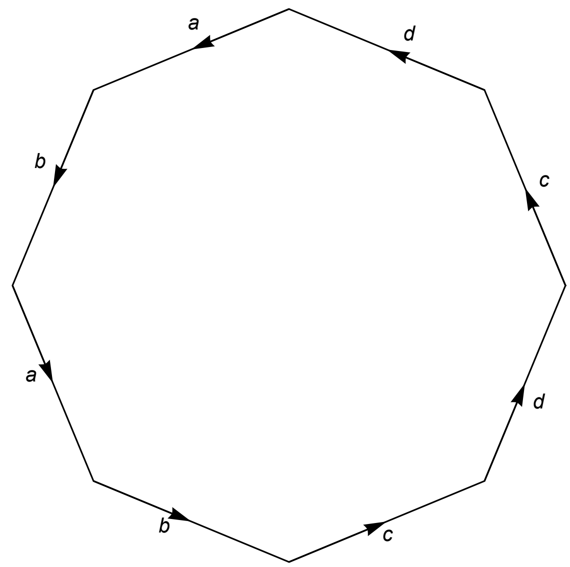

FIGURE 2. Figure for solution to problem 3

to $\mathbb{R}^4 \setminus \{(0,0,0,0)\}$ which defommation retracts onto S^3. $H_k(S^2)$ is 0 except when $k = 0$ and $k = 2$ while $H_k(S^3)$ is 0 except when $k = 0$ and $k = 3$, so S^2 and S^3 cannot be homotopy equivalent, which leads to a contradiction.

7. Recall that if X is a space, the suspension of X is the space SX formed as the quotient of $[-1,1] \times X$ by the relation that collapses both $\{-1\} \times X$ and $\{1\} \times X$ to points. Prove that for all k there are isomorphisms $\tilde{H}_{k+1}(SX) \cong \tilde{H}_k(X)$, where $\tilde{H}_k$ denotes kth reduced homology.

Solution: Regard SX as a quotient of the cone CX, which is itself formed from $[-1,1] \times X$ by collapsing $\{-1\} \times X$ to a point. Note that CX is contractible and thus $\tilde{H}_k(CX) = 0$ for all k. Then SX is formed from CX by collapsing $A = \{1\} \times X$ to a point. Also note that (CX, A) is a good pair, since the product structure gives an obvious neighborhood of A that deformation retracts onto A. Thus $H_k(CX, A) \cong H_k(CX/A, A/A) \cong \tilde{H}_k(CX/A = SX)$. Now, since $H_k(CX)$ is contractible, the long exact sequence for the pair (CX, A) becomes a sequence of isomorphisms $H_k(CX, A) \cong \tilde{H}_{k-1}(A)$ for all k. Since $A \cong X$ we get the desired isomorphisms.

8. Let X be the standard 3–simplex and let A be the 1–skeleton of X. Compute the relative homology $H_k(X, A)$ for all k.

Solution: We will use the long exact sequence for the pair (X, A) and note that $H_k(X) = 0$ except when $k = 0$, $H_0(X) = \mathbb{Z}$ (because X is contractible), $H_k(A) = 0$ except when $k = 0$ and 1, $H_0(A) = \mathbb{Z}$ and $H_1(A) = \mathbb{Z}^3$ (because A is a graph homotopy equivalent to a wedge of 3 circles). Thus for $k > 2$, we have $0 = H_k(X) \to H_k(X, A) \to H_{k-1}(A) = 0$ and thus $H_k(X, A) = 0$. For $k = 2$ we have $0 = H_2(X) \to H_2(X, A) \to H_1(A) = \mathbb{Z}^3 \to H_1(X) = 0$ and thus $H_2(X, A) = \mathbb{Z}^3$. For $k = 1$ we have $0 = H_1(X) \to H_1(X, A) \to H_0(A) = \mathbb{Z} \to H_0(X) = \mathbb{Z}$ but we know that this last inclusion-induced map is an isomorphism and thus $H_1(X, A) = 0$. Lastly $H_0(X, A) = 0$ because of this same isomorphism (or because $H_0(X, A)$ is always 0 when each path component of X contains a path component of A).