Complex Analysis Qualifying Examination

Fall 2023

All problems are of equal weight. Do the easier ones first. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

Notation: $\mathbb{D}, \mathbb{H}$ denote respectively the open unit disc centered at origin and the open upper half plane.

1. Cayley transform $F(z) = \frac{i-z}{i+z}$ is a conformal map from $\mathbb{H}$ to $\mathbb{D}$.

   (a) Show that $F : \mathbb{H} \to \mathbb{D}$, one-to-one and onto, and maps the real line to the unit circle.

   (b) Find the images of the real line $z = x + ib$ for a fixed $b > 0$ and $-\infty < x < \infty$ and the ray $z = a + iy$ for a fixed $a \in \mathbb{R}$ and $y > 0$. You are required to write down the exact equations of these images.

Solution: (a) simple and routine
(b) Compute $F(x + ib)$ and $F(a + iy)$ for $b > 0$ and $a \in \mathbb{R}$ fixed:

$$\begin{aligned} F(x+ib) &= \frac{i-x-ib}{i+x+ib} = \frac{-x+i(1-b)}{x+i(1+b)} = \frac{[-x+i(1-b)][x-i(1+b)]}{[x+i(1+b)][x-i(1+b)]} \\ &= \frac{(-ib-x+i)(-ib+x-i)}{x^2+(1+b)^2} = \frac{-b^2-(x-i)^2}{x^2+(1+b)^2} \\ &= \frac{1-b^2-x^2}{x^2+(1+b)^2} + \frac{2x}{x^2+(1+b)^2} i := u(x) + iv(x) \end{aligned}$$

Then we note that

$$u + 1 = \frac{2+2b}{x^2+(1+b)^2} \quad \text{and} \quad (u+1)^2 + v^2 = \frac{4}{x^2+(1+b)^2}.$$

This yields

$$\frac{1}{x^2+(1+b)^2} = \frac{1}{4}[(u+1)^2 + v^2]$$

and

$$u + 1 = \frac{2+2b}{x^2+(1+b)^2} = \frac{1+b}{2}[(u+1)^2 + v^2].$$

Completing the square in the above to get

$$(u + 1 - \frac{1}{1+b})^2 + v^2 = \frac{1}{(1+b)^2}.$$

This is the circle centered at $(-\frac{b}{1+b}, 0)$ and the radius $\frac{1}{1+b}$.

$$
\begin{aligned}
F(a+iy) &= \frac{i-a-iy}{i+a+iy} = \frac{-a+i(1-y)}{a+i(1+y)} = \frac{[-a+i(1-y)][a-i(1+y)]}{[a+i(1+y)][a-i(1+y)]} \\
&= \frac{(-iy-a+i)(-iy+a-i)}{a^2+(1+y)^2} = \frac{-y^2-(a-i)^2}{a^2+(1+y)^2} \\
&= \frac{1-y^2-a^2}{a^2+(1+y)^2} + \frac{2a}{a^2+(1+y)^2} i := u(y)+iv(y)
\end{aligned}
$$

Then we note that

$$
u+1 = \frac{2+2y}{a^2+(1+y)^2} \quad \text{and} \quad (u+1)^2+v^2 = \frac{4}{a^2+(1+y)^2}.
$$

This yields

$$
\frac{1}{a^2+(1+y)^2} = \frac{1}{4}[(u+1)^2+v^2]
$$

and

$$
v = \frac{2a}{a^2+(1+y)^2} = \frac{a}{2}[(u+1)^2+v^2].
$$

Completing the square in the above to get

$$
(u+1)^2 + (v-\frac{1}{a})^2 = \frac{1}{a^2}.
$$

This is the part of circle centered at $(-1, \frac{1}{a})$ and the radius $\frac{1}{|a|}$ inside the unit disc. When $a=0$, the circle reduces to $v=0$ and the image is the u-axis inside the unit disc. □

2. This question is about Jordan Lemma and its application.

   (a) If Γ_R is the semicircle $z(\theta) = Re^{i\theta}$ with $0 \le \theta \le \pi$. $P(z)$ and $Q(z)$ are polynomials with Degree$(P) \le$ Degree$(Q) - 1$. Show that

   $$
   \lim_{R\to\infty} \int_{\Gamma_R} \frac{P(z)}{Q(z)} e^{iz} dz = 0.
   $$

   (b) Let $a, \lambda > 0$. Compute the integral

   $$
   \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x \sin(\lambda x)}{x^2+a^2} dx
   $$

Solution: (a) The proof of Jordan lemma is to test basic skill in estimation. We assume $\deg(P) = m$ and $\deg(Q) = n$. There exists some constant K so that

$$
\begin{aligned}
\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} e^{iz} dz \right| &= \left| \int_0^\pi \frac{P(Re^{i\theta})}{Q(Re^{i\theta})} e^{iR(\cos\theta + i\sin\theta)} iRe^{i\theta} d\theta \right| \\
&\le KR^{m-n+1} \int_0^\pi e^{-R\sin\theta} d\theta \\
&= KR^{m-n+1} \int_0^{\pi/2} e^{-R\sin\theta} d\theta + KR^{m-n+1} \int_{\pi/2}^{\pi} e^{-R\sin\theta} d\theta
\end{aligned}
$$

For the second integral, we let $\theta = \pi - \phi$ and it is reduced to

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta} d\theta = \int_{0}^{\pi/2} e^{-R\sin\phi} d\phi$$

Hence we have

$$\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} e^{iz} dz \right| \leq 2KR^{m-n+1} \int_{0}^{\pi/2} e^{-R\sin\theta} d\theta.$$

Next we note that $\sin\theta \leq \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \pi/2$, we have

$$\int_{0}^{\pi/2} e^{-R\sin\theta} d\theta \leq \int_{0}^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{2R}(1 - e^{-R})$$

Therefore we have

$$\left| \int_{\Gamma_R} \frac{P(z)}{Q(z)} e^{iz} dz \right| \leq \pi K R^{n-m}(1 - e^{-R})$$

Since $n - m \leq 0$, we have the limit goes to 0 as $R \to \infty$

For (b), we just apply the Jordan lemma and the Cauchy integral formula get

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{xe^{i\lambda x}}{x^2 + a^2} dx + \frac{1}{2\pi i} \int_{\Gamma_R} \frac{ze^{i\lambda z}}{z^2 + a^2} dz = \text{Res}\left(\frac{ze^{i\lambda z}}{z^2 + a^2} \right) \Big|_{z=ia} = \frac{iae^{i\lambda ia}}{2ia} = \frac{1}{2} e^{-a\lambda}$$

Hence we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\sin(\lambda x)}{x^2 + a^2} dx = e^{-a\lambda}$$

□

3. Let f be non-constant and holomorphic in an open set containing the closed unit disc $\bar{\mathbb{D}} = \{z \ |z| \leq 1\}$.

   (a) Show that if $|f(z)| = 1$ when $|z| = 1$, then the image of f contains the unit disc.

   (b) Show that if $|f(z)| \geq 1$ when $|z| = 1$ and there exists a point $z_0 \in \mathbb{D}$ such that $|f(z_0)| < 1$, then the image of f contains the unit disc.

Solution: (a) By Rouché theorem, $f(z)$ and $f(z) - w_0$ have the same number of zeros inside the unit circle provided $|w_0| < 1$. Hence if f has one zero, its image include the unit disc. If f is nonzero, then $\frac{1}{f}$ is holomorphic so $\frac{1}{|f(z)|} \leq 1$ for $z \in \bar{\mathbb{D}}$ by the maximum modulus principle. But then $|f(z)| \geq 1$ for all $z \in \bar{\mathbb{D}}$.which contradicts to the open mapping theorem: Pick any x with $|z| = 1$ then $f(\bar{\mathbb{D}})$ contains a neighborhood of $f(z)$ which include points w with $|w| < 1$ since $|f(z)| = 1$.

(b) Let $w_0 = f(z_0)$ where $|z_0| < 1$ and $|w_0| < 1$. By Rouché theorem again, $f(z)$ and $f(z) - w$ have the same number of zeros for all w with j$|w| < 1$. Since there exists a w (namely w_0) for which $f(z) - w$ has a zero, it has a zero for all $w \in \bar{\mathbb{D}}$. So the image of f contains $\bar{\mathbb{D}}$. □

4. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Show that

(a) $|f(z)| \le |z|$ for all $z \in \mathbb{D}$.

(b) If for some $z_0 \ne 0$, $|f(z_0)| = |z_0|$, then f is a rotation.

(c) $|f'(0)| \le 1$, if equality holds, then f is a rotation.

(d) Let $D(0, R) = \{z : |z| < R\}$. Show that if $F : D(0, R) \to \mathbb{C}$ is holomorphic, with $|F(z)| \le M$ for some M, then

$$\left| \frac{F(z) - F(0)}{M^2 - \overline{F(0)}F(z)} \right| \le \frac{|z|}{MR}.$$

Hint: (a)-(c) is Schwarz lemma and you are required to prove it.

Solution: The proof of Schwarz lemma is standard, we refer to Stein's book. For part (d), we will construct a map from $\mathbb{D}$ to $\mathbb{D}$ from F so the condition of Schwarz lemma is satisfied. Since

$$f_0(z) = Rz : \mathbb{D} \to D(0, R), \quad F : D(0, R) \to D(0, M), \quad f_1(z) = \frac{F(z)}{M} : D(0, R) \to \mathbb{D}$$

and we can composite these maps with with $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ to get

$$f(z) = \psi_\alpha\left(\frac{F(Rz)}{M}\right) : \mathbb{D} \to \mathbb{D}$$

We choose $\alpha = F(0)/M$ so that $f(0) = \psi_\alpha(F(0)/M) = 0$. Apply the Schwarz lemma we have

$$\left| \frac{\frac{F(0)}{M} - \frac{F(Rz)}{M}}{1 - \frac{\overline{F(0)}}{M}\frac{F(Rz)}{M}} \right| \le |z| \quad \Longleftrightarrow \quad \left| \frac{F(0) - F(Rz)}{M^2 - \overline{F(0)}F(Rz)} \right| \le \frac{|z|}{M}$$

for $|z| \le 1$. Then we replace z by $\frac{z}{R}$ and obtain

$$\left| \frac{F(0) - F(z)}{M^2 - \overline{F(0)}F(z)} \right| \le \frac{|z|}{MR} \quad \text{for } |z| < R$$

$\square$

5. Let $G = \mathbb{D} \setminus [\frac{1}{2}, 1)$. Find a bijective conformal map from G to the upper half plane.

Solution: The linear fraction map

$$z_1 = f(z) = i\frac{1 - z}{1 + z} \quad \text{takes } 0 \to i,\ 1 \to 0,\ -1 \to \infty.$$

Hence the linear fractional map maps the region to the region $D_1 = \mathbb{H} - (0, \frac{i}{3}]$ Then $z_2 = z_1^2$ will map D_1 to $D_2 = \mathbb{C} \setminus [-\frac{1}{9}, \infty)$, then the translation $z_3 = z_2 + \frac{1}{9}$ maps D_2 to $D_3 = \mathbb{C} \setminus [0, \infty)$. Then

$$w = \sqrt{z_3} = \sqrt{z_1^2 + \frac{1}{9}}$$

will transfer D_1 to the upper half plane. Hence the conformal map is

$$w = \sqrt{-\left(\frac{1-z}{1+z}\right)^2 + \frac{1}{9}} = \frac{2i}{3(1+z)}\sqrt{(2z-1)(z-2)}.$$

$\square$

6. Suppose that $F(z)$ is holomorphic on a neighborhood U of z_0 with $F(z_0) = F'(z_0) = 0$ and $F''(z_0) =\neq 0$. Prove that there are two curves Γ_1 and Γ_2 passing through z_0, are orthogonal at z_0, $F|_{\Gamma_1}$ is real and has a minimum at z_0 and $F|_{\Gamma_2}$ is also real and has a maximum at z_0. Here $F|_\Gamma$ mean the restriction of F to Γ.

Solution: Let $F(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n$. Then we have $a_0 = a_1 = 0$,$F''(z_0) = 2a_2 \neq 0$ and we can write

$$F(z) = (z-z_0)^2\left[a_2 + a_3(z-z_0) + \cdots\right] = [f(z)]^2.$$

i.e. $f(z)$ is the square-root of $F(z)$. Since $a_2 \neq 0$, $f(z)$ is well defined near z_0 and $f(z) = (z-z_0)h(z)$ with $h(z_0) \neq 0$ and $f'(z_0) = h(z_0)$ by the product rule. Then $w = f(z) = u(z) + iv(z)$ is locally one to one is a disc $Dr(z_0) = \{z,\ |z-z_0| < r\}$ and we let $g(w) = \sigma(w) + i\tau(w)$ be the inverse to f on $\Omega = f(D_r(z_0))$. so we have

$$g(f(z)) = z \text{ for all } z \in D_r(z_0) \text{ and } f(g(w) = w \text{ for all } w \in f(D_r(z_0)).$$

Let Γ_1 consists of those points in $D_r(z_0)$ with $u(z) = u(z_0) - 0$,i.e.,

$$\Gamma_1 := \{z \in D_r(z_0) : u(z) = 0\} = \{z \in D_r(z_0) : \mathrm{Re}(f(z)) = 0\} = \{g(w) : \mathrm{Re}(w) = 0,\ w \in \Omega\}$$

So Γ_1 is the range of function $g(w)$ on the set $\{w \in \Omega :\ \mathrm{Re}(w) = 0\}$ and hence a smooth arc in $D_r(z_0)$. Likewise Γ_2 consists of those points in $D_r(z_0)$ with $v(z) = v(z_0) - 0$,i.e.,

$$\Gamma_2 := \{z \in D_r(z_0) : v(z) = 0\} = \{z \in D_r(z_0) : \mathrm{Im}(f(z)) = 0\} = \{g(w) : \mathrm{Im}(w) = 0,\ w \in \Omega\}.$$

We wish to show that Γ_1 and Γ_2 meet a right angle at z_0. since $g(w)$ is conformal and lines $\mathrm{Re}(w) = u(z)$ and $\mathrm{Im}(w) = v(z) = 0$ meets a right angle at 0. This implies Γ_1 and Γ_2 meet at right angle at z_0. Direct computation yields the same result: Let $w = s + it$ and $g(w) = \sigma(s,t) + i\tau(s.t)$. Then

$$\nabla\sigma \cdot \nabla\tau = \frac{\partial\sigma}{\partial s}\frac{\partial\tau}{\partial s} + \frac{\partial\sigma}{\partial t}\frac{\partial\tau}{\partial t}$$

But g is holomorphic and it satisfies the Cauchy-Riemann equations

$$\frac{\partial\sigma}{\partial s} = \frac{\partial\tau}{\partial t} \quad \text{and} \quad \frac{\partial\sigma}{\partial t} = -\frac{\partial\tau}{\partial s}$$

hence $\nabla\sigma \cdot \nabla\tau = 0$. Finally

$$F(z)|_{\Gamma_1} = (f(z))^2|_{\Gamma_1} = (iv(z))^2 = -v^2(z),\ F(z)|_{\Gamma_2} = (f(z))^2|_{\Gamma_2} = (u(z))^2$$

and $F|_{\Gamma_1}$ has a local maximum at z_0 and $F|_{\Gamma_2}$ has a local minimum at z_0. $\square$