Complex Analysis Qualifying Exam — Spring 2023

Show work and carefully justify/prove your assertions. For example, if you use a theorem that has a name, mention the name. Arrange your solutions in numerical order even if you do not solve them in that order.

1. (10 points) Let z_k ($k = 1, \cdots, n$) be complex numbers lying on the same side of a straight line passing through the **origin** but not on the line. Show that

$$z_1 + z_2 + \cdots + z_n \neq 0, \quad 1/z_1 + 1/z_2 + \cdots + 1/z_n \neq 0.$$

Hint: Consider a special situation first.

2 (10 points) Suppose f is an entire function whose range omits numbers in $(-\infty, 0]$. Prove that f is constant. Hint: The analytical function $g(z) := \sqrt{z}$ maps the region $\mathbb{C} \backslash (-\infty, 0]$ to the open right half plane.

3. (10 points) (1) Show that the series

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

defines an analytic function in the region $x > 1$, where $z = x + iy$ and $n^z = e^{z \log n}$.

(2) Find series representation of $\zeta^{(k)}(z)$ ($k \geq 1$) in $x > 1$ and justify you answer.

4. (10 points) Let γ be a piecewise smooth simple closed curve oriented counterclockwise with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \, d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

5. (10 points) Use complex analysis to compute the following integral

$$\int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx$$

6. (10 points) Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic on the unit disc and has a zero of order n at 0. Prove that $|f(z)| \leq |z^n|$ for all $z \in \mathbb{D}$ and obtain a sharp upper bound on $|f^{(n)}(0)|$.

7. (10 points) Find the fractional linear transformation that maps the circle $|z| = 2$ into $|z + 1| = 1$, the point -2 into the origin, and the origin into i.