By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): Solutions ____________ Name (print): ____________

Student Number: ____________

Instructor's Name: ____________ Class Time: ____________

Problem Number	Points Possible	Points Made
1	25	
2	20	
3	10	
4	10	
5	15	
6	5	
7	10	
8	10	
9	20	
10	15	
11	10	
12	10	
13	10	
Total:	170	

- If you need extra space use the last page.
- Please show your work. **An unjustified answer may receive little or no credit.**
- If you make use of a theorem to justify a conclusion then state the theorem used by name.
- Your work must be **neat**. If I can't read it (or can't find it), I can't grade it.
- The total number of possible points that is assigned for each problem is shown here. The number of points for each subproblem is shown within the exam.
- Please turn off your mobile phone.
- You are only allowed to use a TI-30 calculator. No other calculators are permitted.
- A calculator is not necessary, but numerical answers should be given in a form that can be directly entered into a calculator.

1. Determine the first derivative of each of the following functions. Print your answer in the box provided.

   (a) [5 pts] $f(x) = e^{3x} + 5x^2 + 2001$.

   $f'(x) = 3e^{3x} + 10x$

   (b) [5 pts] $h(x) = x^2 \cos(x) + 42$.

   $h'(x) = x^2 \cdot -\sin(x) + \cos(x) \cdot 2x$

   (c) [5 pts] $G(r) = \dfrac{r+4}{3r^2+7}$. $= (r+4)(3r^2+7)^{-1}$

   $G'(r) = \dfrac{(3r^2+7)(1) - (r+4)(6r)}{(3r^2+7)^2}$ <u>or</u> $(r+4)(-1(3r^2+7)^{-2} \cdot 6r) + (3r^2+7)^{-1} \cdot 1$

(d) [5 pts] $M(x) = \arctan(3x)$.

$M'(x) = \dfrac{1}{1+(3x)^2} \cdot 3$

(e) [5 pts] $Q(s) = (1+s)^{\frac{1}{s}}$.

$$\ln(Q(s)) = \frac{1}{s}\ln(1+s)$$

$$\frac{1}{Q(s)} \cdot Q'(s) = \frac{1}{s} \cdot \frac{1}{1+s} + \ln(1+s) \cdot \frac{-1}{s^2}$$

$$Q'(s) = Q(s)\left[\frac{1}{s(1+s)} - \frac{\ln(1+s)}{s^2}\right]$$

other method:

$$Q(s) = e^{\ln((1+s)^{1/s})}$$

$$Q(s) = e^{\frac{1}{s}\ln(1+s)}$$

$$Q'(s) = e^{\frac{1}{s}\ln(1+s)}\left(\frac{1}{s} \cdot \frac{1}{1+s} - \frac{\ln(1+s)}{s^2}\right)$$

$$Q'(s) = (1+s)^{1/s}\left(\frac{1}{s} \cdot \frac{1}{1+s} - \frac{\ln(1+s)}{s^2}\right)$$

$Q'(s) = (1+s)^{1/s}\left[\dfrac{1}{s(1+s)} - \dfrac{\ln(1+s)}{s^2}\right]$

2. Determine **the most general anti-derivative** indicated in each of the following expressions. Print your answer in the box provided.

______ (a) [5 pts] $\int (5x + 3)\ dx$.

Anti-Derivative: $\frac{5}{2}x^2 + 3x + C$

______ (b) [5 pts] $\int \left(\cos(4x) + e^{2x} + 9\right)\ dx$.

Anti-Derivative: $\frac{1}{4}\sin(4x) + \frac{1}{2}e^{2x} + 9x + C$

Points earned: ______
out of a possible 10 points

(c) [5 pts] $\int \frac{x}{3-x^2}\,dx. = \int \frac{1}{3-x^2} \cdot x\,dx = \frac{-1}{2}\int \frac{1}{3-x^2} \cdot -2x\,dx$

$u = 3 - x^2$
$du = -2x\,dx$

$= \frac{-1}{2}\int \frac{1}{u}\,du$

$= -\frac{1}{2}\ln|u| + C$

Anti-Derivative: $-\frac{1}{2}\ln|3-x^2| + C$

(d) [5 pts] $\int \sin(x)\cos(x)\,dx. = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C$

$u = \sin(x)$
$du = \cos(x)dx$

or $\int \sin(x)\cos(x)dx = -\int \cos(x) \cdot -\sin(x)dx = -\int u\,du$
$= -\frac{1}{2}u^2 + C = \frac{-1}{2}\cos^2(x) + C$

$u = \cos(x)$
$du = -\sin(x)dx$

or $\int \sin(x)\cos(x)dx = \int \frac{1}{2}\sin(2x)dx = \frac{-1}{4}\cos(2x) + C$

Anti-Derivative: $\frac{1}{2}\sin^2(x) + C = \frac{-1}{2}\cos^2(x) + C = \frac{-1}{4}\cos(2x) + C$

3. Determine the value of each of the following limits. Indicate if a limit approaches ∞ or $-\infty$ otherwise print DNE if the limit does not exist. **Show all of your work and justify your conclusions.**

(a) [5 pts] $\lim_{x\to 0} \frac{\sin^2(x)}{1-\cos(x)}$. $\underset{LH}{=} \lim_{x\to 0} \frac{2\cancel{\sin(x)}\cos(x)}{\cancel{\sin(x)}} = \boxed{2}$

$\frac{0}{0}$

<u>or</u> $\lim_{x\to 0} \frac{\sin^2(x)}{1-\cos(x)} = \lim_{x\to 0} \frac{1-\cos^2(x)}{1-\cos(x)} = \lim_{x\to 0} \frac{(1+\cos(x))\cancel{(1-\cos(x))}}{\cancel{1-\cos(x)}} = \boxed{2}$

(b) [5 pts] $\lim_{x\to 1} f(x)$ where

$$f(x) = \begin{cases} x+2 & x \le 1, \\ 1 & x > 1. \end{cases}$$

Check both one-sided limits:

$$\left.\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} (1) = 1 \\ \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} (x+2) = 3 \end{aligned}\right\}$$

These are not equal so $\lim_{x\to 1} f(x)$ <u>does not exist</u>.

4. Use the following tables to answer each of the questions below.

x	1	2	3	4	5	6	7	8
$f(x)$	4	3	5	8	6	7	1	2
$f'(x)$	3	2	1	6	4	5	8	7

x	1	2	3	4	5	6	7	8
$g(x)$	6	5	1	2	8	7	4	3
$g'(x)$	1	4	2	9	3	6	8	5

______ (a) [5 pts] Determine the value of $p'(2)$ where $p(x) = f(x) \cdot g(x)$.

$p'(x) = f(x)g'(x) + g(x) \cdot f'(x)$

$p'(2) = f(2)g'(2) + g(2) \cdot f'(2)$

$= 3 \cdot 4 + 5 \cdot 2$

$= 22$

$p'(2) =$ 22

______ (b) [5 pts] Determine the equation for the tangent line of $p(x) = f(x) \cdot g(x)$ at $x = 2$.

$p'(2) = 22$

$p(2) = f(2) \cdot g(2) = 3 \cdot 5 = 15$

$y - y_1 = m(x - x_1)$

$y - 15 = 22(x - 2)$

or

$y = mx + b$

$y = 22x + b$

$15 = 22(2) + b$

$b = 15 - 44 = -29$

$y = 22x - 29$

Tangent Line: $y - 15 = 22(x - 2)$ or $y = 22x - 29$

5. The following questions refer to the limit definition of the derivative.

(a) [5 pts] State the limit definition of the derivative of a function, $f(x)$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(b) [10 pts] Use the **limit definition of the derivative** to show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{(x+1)^2}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{x+h}{x+h+1}-\frac{x}{x+1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{x+h}{x+h+1}\cdot\frac{x+1}{x+1}-\frac{x}{x+1}\cdot\frac{x+h+1}{x+h+1}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{x^2+x+xh+h}{(x+h+1)(x+1)}-\frac{x^2+xh+x}{(x+h+1)(x+1)}\right)$$

$$=\lim_{h\to 0}\frac{1}{\cancel{h}_1}\left(\frac{\cancel{h}^1}{(x+h+1)(x+1)}\right)$$

$$=\lim_{h\to 0}\frac{1}{(x+h+1)(x+1)}$$

$$=\boxed{\frac{1}{(x+1)^2}}$$

6. [5 pts] Evaluate

$$\frac{d}{dx}\int_0^{x^2} \frac{\sin(t)}{1+t}\,dt.$$

Chain rule:

$f(x)=\int_0^{x^2} \frac{\sin(t)}{1+t}\,dt$

$g(x)=\int_0^{x} \frac{\sin(t)}{1+t}\,dt$, $h(x)=x^2$ means $f(x)=g(h(x))$

$f'(x)=g'(h(x))\cdot h'(x)$

$g'(x)=\frac{d}{dx}\left(\int_0^x \frac{\sin(t)}{1+t}\,dt\right)=\frac{\sin(x)}{1+x}$

$g'(h(x))=\frac{\sin(x^2)}{1+x^2}$

$h'(x)=2x$

$f'(x)=g'(h(x))\cdot h'(x)=\boxed{\frac{\sin(x^2)}{1+x^2}\cdot 2x}$

7. The efficiency of a Carnot heat engine is defined to be

$$E(T) \quad = \quad 1 - \frac{C}{T},$$

where C is the temperature of the surrounding environment in Kelvin, and T is the temperature of the heat source. For the following questions assume that $C = 300$ Kelvin is a constant.

______ (a) [5 pts] Determine the linearization of $E(T)$ at $T = 500$K.

$E(T) = 1 - \frac{300}{T}$

$E(500) = 1 - \frac{3}{5} = \frac{2}{5}$

$E'(T) = \frac{300}{T^2}$

$L(x) = f(a) + f'(a)(x-a)$

$L(T) = E(500) + E'(500)(T-500)$

$E'(500) = \frac{300}{(500)^2}$

$L(T) = \frac{2}{5} + \frac{300}{(500)^2}(T-500)$

(b) [5 pts] Use the linearization to approximate the change in E if $T = 500 \pm 20$K.

______

$\Delta E \approx \Delta L$

Note: $L(500) = E(500)$

$\Delta L = L(500 \pm 20) - L(500)$

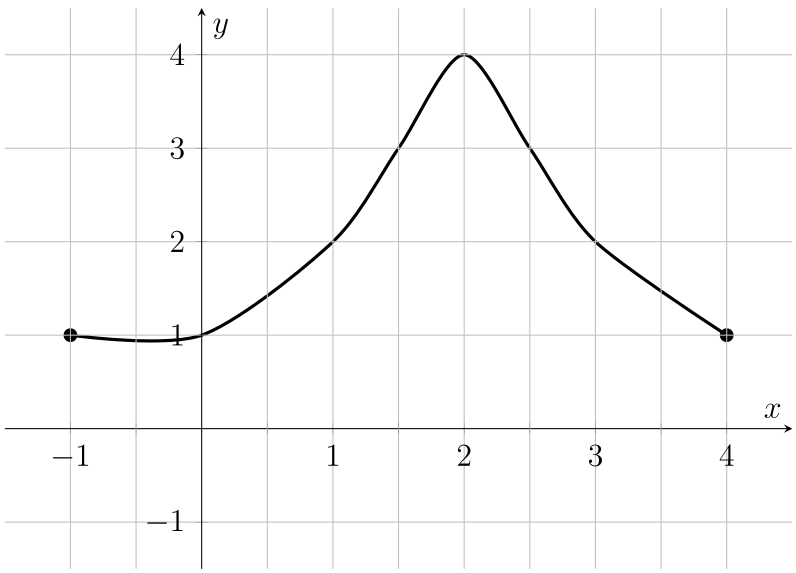
$= \left(\frac{2}{5} + \frac{300}{(500)^2}(\pm 20)\right) - \frac{2}{5} = \boxed{\pm \frac{300 \cdot 20}{500^2}} = \pm \frac{3}{125}$

OR $\Delta E \approx dE$

$dE = \frac{300}{T^2}\, dT$

$dE = \frac{300}{500^2}(\pm 20) = \pm \frac{3}{125}$

8. The graph of a function is given in the plot below. The domain of the function is the closed interval $[-1, 4]$. Use the plot to answer each of the following questions.



_______ (a) [5 pts] Estimate the area under the curve from $x = 1$ to $x = 3$ using a Riemann sum. The Riemann sum should use four rectangles of equal width, and use the left endpoint of each subinterval.

$n=4,\ \Delta x = \frac{3-1}{4} = \frac{1}{2}$

$[1, 1.5][1.5, 2][2, 2.5][2.5, 3]$

$f(1)\cdot\frac{1}{2} + f(1.5)\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} + f(2.5)\cdot\frac{1}{2}$

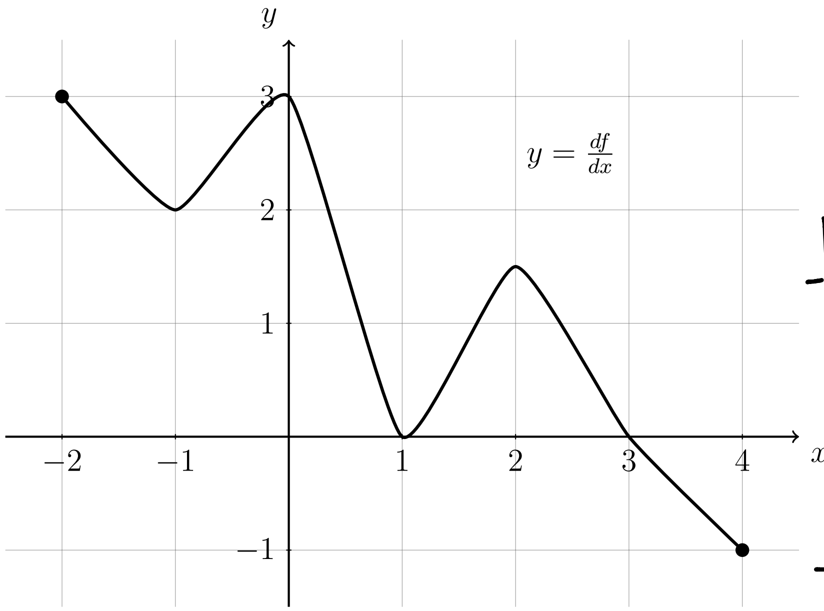
$= 2\cdot\frac{1}{2} + 3\cdot\frac{1}{2} + 4\cdot\frac{1}{2} + 3\cdot\frac{1}{2}$

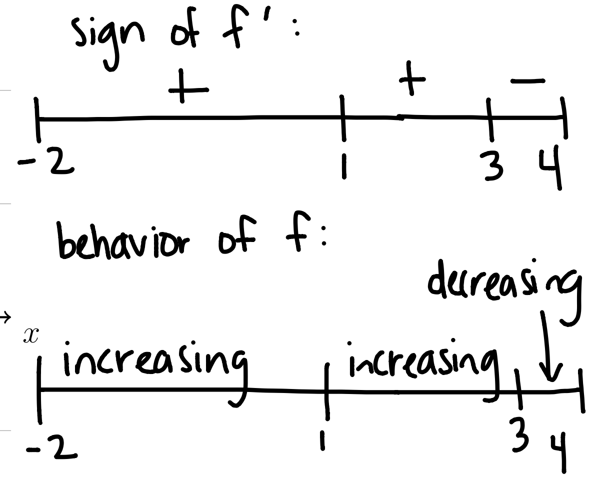
$= \frac{1}{2}(2+3+4+3) = \boxed{6}$

_______ (b) [5 pts] If you use the left endpoints on each subinterval to construct a Riemann sum to estimate the area from $x = 2$ to $x = 4$ will your estimate be more than, equal to, or less than the true value?

Left endpoints will yield an answer that is greater than the true value because the function is decreasing on that interval.

9. The graph of the derivative, $f'(x)$, of a function is given in the plot below, and the domain of the derivative is $[-2, 4]$. Answer each of the following questions.

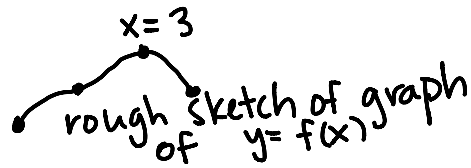




(a) [5 pts] For what values of x, if any, are there local extrema for f? (Indicate which values give local maxima and which give local minima.) Ignore the endpoints, $x = -2$ and $x = 4$.

local max at $x=3$

(ignore local minima at $x=-2,4$)



(b) [5 pts] For what intervals of x, if any, is f increasing?

f is increasing on $(-2,3)$

also ok: $[-2,3]$

(c) [5 pts] For what values of x, if any, does f have an inflection point.

$f''=(f')'$ changes sign at $x=-1$, $x=0$, $x=1$, and $x=2$ since f' changes from increasing to decreasing, or from decreasing to increasing at those values of x.

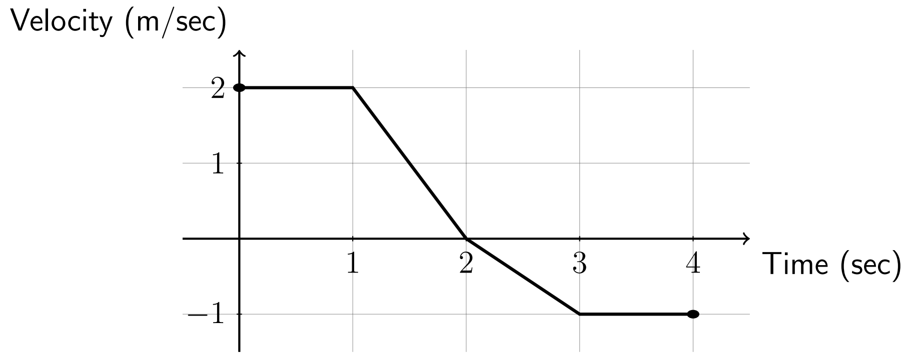
(d) [5 pts] For what intervals of x, if any, is f concave down?

f is concave down when f' is decreasing:

$(-2,-1) \cup (0,1) \cup (2,4)$

ok to use closed intervals instead

10. An object is moving on a straight, level track. The graph of its velocity in meters per second as a function of time in seconds is shown in the plot below. Answer each of the following questions.



(a) [5 pts] The object's start point is its position at $t = 0$ seconds. How far is the object from its start point at the time $t = 4$ seconds?

$$s(4) - s(0) = \int_0^4 v(t)\,dt = 3 - \frac{3}{2} = \boxed{\frac{3}{2} \text{ meters}}$$

(b) [5 pts] Determine the average velocity of the object between $t = 0$ seconds and $t = 4$ seconds.

$$\text{avg. velocity} = \frac{s(4)-s(0)}{4-0} = \frac{1}{4}\int_0^4 v(t)\,dt = \frac{1}{4}\left(\frac{3}{2}\right) = \frac{3}{8}$$

meters per second

(c) [5 pts] What is the total distance traveled from $t = 0$ seconds to $t = 4$ seconds?

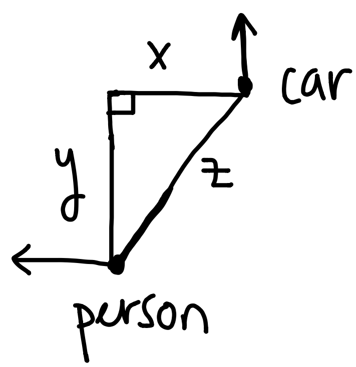
Break down $[0,4]$ into intervals where motion is all in positive direction $(v > 0)$ or all in negative direction $(v < 0)$.

$[0,2]$: $\int_0^2 v(t)\,dt = 3$ units, so 3 units in the positive direction

$[2,4]$: $\int_2^4 v(t)\,dt = -\frac{3}{2}$, or $\frac{3}{2}$ units in the negative direction

The total distance traveled is $\boxed{3 + \frac{3}{2} \text{ meters}} = \frac{9}{2}$ meters.

11. [10 pts] A car is moving North at 65 miles per hour. A person is walking due ~~East~~ West on a different road. Determine how fast the person is moving at the moment when the person is 50 miles West and 70 miles South of the car and the distance between the person and the car is increasing at a rate of 55 miles per hour.



goal: $\frac{dx}{dt}$ (and then speed)

$x = 50$ mi $\quad \frac{dz}{dt} = 55$ mph, $\quad \frac{dy}{dt} = 65$ mph
$y = 70$ mi

$$x^2 + y^2 = z^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$50^2 + 70^2 = z^2$$
$$2500 + 4900 = z^2$$
$$\sqrt{7400} = z$$

$$\frac{dx}{dt} = \frac{z\frac{dz}{dt} - y\frac{dy}{dt}}{x} = \frac{\sqrt{7400}(55) - 70(65)}{50}$$ miles per hour

12. [10 pts] A component of an engine will be connected to a heat sink by a cylindrical rod of length 8 cm with radius r. The rate of heat flow through the rod is given by

$$G = \frac{1}{8}\mu r^2,$$

where $\mu > 0$ is the *thermal conduction coefficient*. The cost to make the rod is equal to the radius plus the thermal conduction coefficient. If $30 is allocated to produce the rod, determine the radius, r, and thermal conduction coefficient, μ, of the rod that will maximize the rate of heat flow.

goal: maximize G

cost $= r + \mu$ so $30 = r + \mu$

$G = \frac{1}{8}\mu r^2 = \frac{1}{8}(30-r)r^2 = \frac{1}{8}(30r^2 - r^3)$

domain: $(0,30)$ since $30 = r + \mu$ and $\mu > 0$, $r > 0$

$G' = \frac{1}{8}(60r - 3r^2) = \frac{3}{8}r(20-r)$

Note: can apply 2nd derivative test instead

critical number: $r = 20$

	$(0,20)$	$(20,30)$
G'	$G'(1) = \frac{3}{8}\cdot 19$	$G'(21) = \frac{3}{8}\cdot 21 \cdot -1$
sign G'	+	—
behavior of G	increasing	decreasing

Since G changes from increasing to decreasing, G has a local max at $r = 20$. Since there is only one critical number, the local max is an absolute max.

Heat flow is maximized when $r = 20$ cm and $\mu = 10$.

13. [10 pts] Find all points, other than (0,0), where the curve

$$x^3 + xy + y^3 \;=\; 0$$

has a vertical tangent line. *Your answer(s) should be in the form of coordinate pairs.*

$\frac{dy}{dx}$ is undefined, and secant slopes approach $\pm\infty$.

$$3x^2 + x \cdot \frac{dy}{dx} + y \cdot 1 + 3y^2 \frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -3x^2 - y$$

$$\frac{dy}{dx}(x+3y^2) = -3x^2 - y$$

$$\frac{dy}{dx} = \frac{-3x^2-y}{x+3y^2}$$

undefined when $x+3y^2=0$

$x = -3y^2$

also: $x^3+xy+y^3=0$

$$(-3y^2)^3 + (-3y^2)y + y^3 = 0$$

$$-27y^6 - 3y^3 + y^3 = 0$$

$$-27y^6 - 2y^3 = 0$$

$$y^3(-27y^3-2) = 0$$

$\cancel{y=0}$ $\cancel{x=0}$

$$-27y^3 = 2$$

$$y = \sqrt[3]{-2/27} = \frac{-\sqrt[3]{2}}{3}$$

$$x = -3\left(-\frac{\sqrt[3]{2}}{3}\right)^2 = \frac{-3}{9}(2)^{2/3} = -\frac{2^{2/3}}{3}$$

$$\boxed{\left(-\frac{2^{2/3}}{3}, \frac{-2^{1/3}}{3}\right)}$$

 Points earned: ____ out of a possible 10 points

Extra space for work. **Do not detach this page.** If you want us to consider the work on this page you should print your name, instructor and class meeting time below.

Name (print): ______________ Instructor (print): ______________ Time: ______________