Qualifying Exam: Complex Analysis — Fall 2017

Justify your answers and state clearly any theorem, proposition or lemma that you are applying. You should not cite examples, exercises, or problems from any source (other than this exam). Cross out the parts you do not want to be graded.

There are seven questions in this exam–answer them all.

1. (14) The straight lines $y = \sqrt{3}x$ and $x + y = 1$ in the z-plane are mapped onto the w-plane (endowed with (u, v) coordinates) by the transformation

$$w = z^2.$$

a) What are the images of these lines in the w-plane? Express your answer analytically, i.e., in terms of the (u, v) coordinates and sketch the images.

b) What is the angle of intersection between the images?

2. (14)

a) Let $f(z)$ be analytic inside and on a circle C of radius r centered at a. Prove that

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

b) Let L be a simple closed curve. Suppose that $f(z)$ is analytic inside and on L. Prove that the *maximum* value of $|f(z)|$ occurs on L, unless $f(z)$ is a constant.

3. (14) Let $\{f_k(z)\}, k = 1, 2, 3, \ldots$ be a sequence of analytic functions in a star-shaped domain $\mathcal{D} \subset \mathbb{C}$. Suppose that the sequence of partial sums

$$F_n(z) := \sum_{k=1}^{n} f_k(z), \quad n = 1, 2, 3, \ldots$$

is uniformly convergent in $\mathcal{D}$ to the limit function $F(z)$. Prove that $F(z)$ is analytic in $\mathcal{D}$.

4. (14) Find Laurent series about the indicated singularity for each of the following functions. Classify the type of singularity in each case and describe the (maximal) domain of convergence of each series:

(1) $\dfrac{z - \sin z}{z^3}$, $z = 0$, and

(2) $\dfrac{z}{(z + 1)(z + 2)}$, $z = -2$.

5. (14) Evaluate

$$\int_0^\infty \frac{\cos 5x}{x^2+1} dx.$$

6. (14) Let $\mathbb{E} \subset \mathbb{C}$ denote the *open unit-disk*. Let $f : \mathbb{E} \to \mathbb{E}$ be an analytic function which is *not* identically zero in some neighborhood of the origin. Prove that:

   a) There exists a positive integer m such that

   $$f(0) = f'(0) = \ldots = f^{(m-1)}(0) = 0 \text{ and } f^{(m)}(0) \neq 0.$$

   b) For this number m, the following hold for all $z \in \mathbb{E}$:

   $$|f(z)| \leq |z|^m \quad \text{and } |f^{(m)}(0)| \leq m!.$$

7. (14) Let L be a positive constant. Let $\{a_n\}_0^\infty$ be a strictly decreasing sequence of positive numbers, where $a_n \geq L$ for all n. Let $\mathbb{E} \subset \mathbb{C}$ denote the *open unit-disk*.

   a) Prove that $f(z) := \sum_{n=0}^\infty a_n z^n$ defines an analytic function in $\mathbb{E}$.

   b) What is the radius of convergence of the power series $f(z)$?

   c) Show that f has no zeros in $\mathbb{E}$.

Good Luck!