QUALIFYING EXAMINATION IN COMPLEX ANALYSIS

August 15, 2007

12:00–2:00 pm

As usual, $\mathbb{D}$ denotes the (open) unit disk and $\mathbb{H}$ the upper half-plane. Provide justifications as appropriate.

1. (20 points) Use methods of complex analysis to evaluate

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx\,.$$

2. (10 points) Let $f(z) = \dfrac{z+2}{z^2+z}$. Give the Laurent expansion of f that converges on
   a. $\{z : 0 < |z| < 1\}$
   b. $\{z : 1 < |z+1|\}$

3. (10 points) Let $n \in \mathbb{N}$. Prove that the equation $e^z = az^n$ has n solutions in $\mathbb{D}$ if $|a| > e$ and none if $|a| < 1/e$.

4. (25 points)
   a. (5 pts) State the Schwarz reflection principle (the "standard" version involving reflection across the real axis).
   b. (10 pts) Give (with justification) a linear fractional transformation T mapping $\mathbb{D}$ to $\mathbb{H}$. Let $g(z) = \overline{z}$; show that $(T^{-1}\circ g\circ T)(z) = 1/\overline{z}$.
   c. (10 pts) Suppose f is holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and real on the unit circle C. Prove that f must be constant.

5. (20 points) Suppose $\{f_n\}$ is a sequence of analytic functions on $\mathbb{D}$ that converges uniformly on compact subsets to f. Prove carefully that f is analytic on $\mathbb{D}$ and that

$$\text{if} \quad f_n(z) = \sum_{j=0}^{\infty} a_j^{(n)} z^j\,, \quad \text{then} \quad f(z) = \sum_{j=0}^{\infty} a_j z^j\,,$$

where $a_j = \lim_{n\to\infty} a_j^{(n)}$ for all $j = 0, 1, 2, \ldots$.

6. (15 points) Suppose f is holomorphic on a region Ω, $a \in \Omega$, and $f'(a) \neq 0$. Being sure to check all hypotheses and state the theorem carefully, apply the (real) inverse function theorem to prove that f has a holomorphic local inverse on a neighborhood of $b = f(a)$. (Hint: You will want to use some form of the Cauchy Riemann equations at least once.)