Complex Analysis Qualifying Examination

Spring 2019

All problems are of equal weight. Arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

Notation: $\mathbb{D}$ denotes the open unit disk and $\mathbb{H}$ denotes the open upper half plane.

1. Define
$$E(z) = e^x(\cos y + i \sin y)\ .$$
(i) Show that $E(z)$ is the unique function analytic on $\mathbb{C}$ that satisfies
$$E'(z) = E(z), \quad E(0) = 1.$$
(ii) Conclude from (i) that $E(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

2. Let $0 < a < 3$. Evaluate $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^3} dx$ using complex analysis.

3. Let $R > 0$. Suppose f is holomorphic on $\{z \mid |z| < 3R\}$. Let
$$M_R := \sup_{|z|\le R} |f(z)|, \quad N_R := \sup_{|z|\le R} |f'(z)|$$
(i) Estimate M_R in terms of N_R from above.

   (ii) Estimate N_R in terms of M_{2R} from above.

4. Let f be a meromorphic function on the complex plane with the property that $|f(z)| \le M$ for all $|z| > R$, for some constants $M > 0,\ R > 0$.

   Prove that $f(z)$ is a rational function, i.e., there exist polynomials p, q so that $f = \frac{p}{q}$.

5. Suppose that $f(z)$ is holomorphic on $\mathbb{D}$ and that $|f(z)| < 1$. Show that
$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)} \right| \le \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|$$
Hint: Schwarz lemma.

6. Find a conformal map from $\{z \mid |z - 1/2| > 1/2,\ \mathrm{Re}(z) > 0\}$ to $\mathbb{H}$.

7. Let $\Omega \subset \mathbf{C}$ be a connected open subset. Let $\{f_n : \Omega \to \mathbf{C}\}_{n=1}^{\infty}$ be a sequence of holomorphic functions uniformly bounded on compact subsets of Ω. Let $f : \Omega \to \mathbf{C}$ be a holomorphic function such that the set $\{z \in \Omega \mid \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.