Preliminary Exam in Complex Analysis

Fall 1992

1. Find a conformal map of the unit disk $\Delta=\{z:|z|<1\}$ to the first quadrant $Q=\{z=x+iy: x>0, y>0\}$ that sends $z=0$ to $z=1+i$.

2. The principal determination of $z=\arctan w$ is the solution of $\tan z=w$ for which $-\pi/2 < \text{Re} z \leqq \pi/2$. Determine the domain of this principal determination and an expression for the principal determination of $\arctan w$ in terms of a principal determination of the logarithm function.

3. Let $w=f(z)$ be a function that is analytic on the closure of the unit disc Δ in the complex plane. Assume that f is not identically zero and show there are points $z_1, \ldots, z_N$ in Δ and numbers $r_1, \ldots, r_N$ so that

$$u(z)=\log|f(z)| - \sum_{j=1}^{N} r_j \log|z - z_j|$$

is harmonic on $\Delta\setminus\{z_1, \ldots, z_N\}$.

4. Prove that a non-constant analytic function is an open mapping.

5. Determine (for all values in the domain of F) the value of

$$F(w)=\int_C \frac{1}{1-wz^2}dz,$$

where C is the positively oriented unit circle $|z|=1$.

6. Evaluate

$$\frac{1}{2\pi i}\int_C z \frac{f'(z)}{f(z)}dz$$

over a (positively oriented) large circle if f(z) is a polynomial.

7. Give an explicit expression for a meromorphic function $w=f(z)$ defined on the complex plane whose only poles are simple poles at each point in the set

$$Z+iZ=\{\omega_{m,n}=m+in : m,n\in Z\}$$

and such that the residue of f is one at each pole. Be sure to state carefully any results used in your construction.

8. Show the function in Problem 7 cannot be doubly periodic. That is show it is impossible that $f(z+\omega_{m,n})=f(z)$, for all m,n in Z.

9. Let g be analytic on a disc in the complex plane. Suppose that the differential equation

$$\frac{dy}{dz} = yg(z)$$

has an analytic solution in a neighborhood of each point in this disc. Show there is a global analytic solution to this differential equation on this disc.

10. Let $g=g(\zeta)$ be continuous on $|\zeta|=1$ and define

$$G(z)=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{g(\zeta)}{\zeta-z}d\zeta, \quad |z|\neq 1.$$

Determine for $|u|=1$

$$\lim_{r\to 1^+}[G(ru)-G(r^{-1}u)].$$