Complex Analysis Qualifying Examination — Fall 2003

Show work and carefully justify/prove your assertions.

Problems

1. Let $a > 0$. Evaluate $\int_0^\infty \frac{\cos ax}{1+x^2}\,dx$ using the methods of complex analysis. Justify all steps.

2. Let D be the region obtained by removing the interval $(-1, 0]$ from the disk $|z| < 1$. Find a conformal map from D on to the unit disk.

3. Let $\dfrac{1}{z^2 \tan z} = \displaystyle\sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $\dfrac{1}{z^2 \tan z}$ in $0 < |z| < \pi$. Find the principal part $\sum_{n=-\infty}^{-1} a_n z^n$.

4. Let $f(z)$ be entire and assume that

$$\lim_{z \to \infty} \frac{f(z)}{z^n} = 0,$$

where $n \geq 1$ is an integer. Show that $f(z)$ is a polynomial in z of degree $\leq n - 1$.

5. Let $|a_k| < 1$ $(k = 1, 2, \cdots, n)$, $|b| < 1$ and

$$f(z) = \frac{z - a_1}{1 - \bar{a}_1 z} \frac{z - a_2}{1 - \bar{a}_2 z} \cdots \frac{z - a_n}{1 - \bar{a}_n z}.$$

Show that $f(z) = b$ has n solutions in $|z| < 1$.