COMPLEX ANALYSIS PRELIMINARY EXAMINATION, SPRING 1999

Show your work and justify all your reasoning.

1. (a) Give the power series expansion about $z = 1$ of the branch of $f(z) = z^i$ with $f(1) = 1$. Find its radius of convergence.
(b) Find all solutions of the equation $z^i = i$.

... 2. Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ a holomorphic function. Show, using only advanced calculus, that

$$\int_\gamma f(z)\, dz = 0$$

for any smooth closed loop $\gamma \subset U$.

3. Prove that there is no meromorphic function f on $\mathbb{C}$ such that $f(x) = \arctan x$ for all $x \in \mathbb{R}$.

4. Give the Laurent expansion of $\dfrac{1}{z(z-1)}$ in (a) the annulus $\{0 < |z| < 1\}$ and (b) the annulus $\{1 < |z| < 2\}$.

5. Compute

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}\, dx.$$

6. Give a formula for a conformal mapping from the region

$$U := \{re^{i\theta} : 0 < r < 1, 0 < \theta < \frac{\pi}{4}\}$$

onto the unit disk.

7. Suppose $f(z)$ is a meromorphic function on $\mathbb{C}$ such that $\lim_{z\to\infty} |f(z)|$ exists (possibly taking the value ∞). Show that f is a rational function.

8. Let $U \subset \mathbb{C}$ be a domain and $F : [0,1] \times U \to \mathbb{C}$ a bounded continuous function such that $z \mapsto F(t,z)$ is holomorphic on U for every $t \in [0,1]$. Define $f : U \to \mathbb{C}$ by

$$f(z) = \int_0^1 F(t,z)\, dt.$$

Prove that f is holomorphic.

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