Complex Analysis Qualifying Examination — Spring 2004

Show work and carefully justify/prove your assertions.

1. Let $a > 0$. Evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx$ using the method of complex analysis. Justify all steps.

2. Assume that $f(z)$ is a non-constant function that is analytic in $\mathbb{C}$ except for poles. Show that for any $R > 0$, the number of poles in $|z| < R$ is finite.

3. Let $G = \{z : |z-1| < \sqrt{2},\ |z+1| < \sqrt{2}\} \backslash [0, i)$. Find a bijective conformal map from G to the upper half plane.

4. Let $f(z)$ be analytic on the complex plane. Prove that $f(z)$ is necessarily a constant if $f(\bar{z})$ is also analytic.

5. Let f be analytic on a bounded domain D, and continuous non-zero on the closure $\overline{D}$. Show that $f(z) = e^{i\theta}M$ (where θ is a real constant) if $|f(z)| = M$ (a constant) for $z \in \partial D$, the boundary of D.