Num Ana

Ph.D. Preliminary Examination
Numerical Analysis
September 22, 1994

Instructions: Do any seven and only seven of the following eight problems. Strike out the single problem on the examination that you do not wish to have graded. Please start each problem on a new sheet of paper, write on only one side of the paper, and number and sign your name on the top of each page. Turn in only seven problems.

1. Determine the LU factorization (with L unit lower triangular) of the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 4 \end{bmatrix},$$

and then verify that your computed LU factorization for A is correct. (Use exact rational arithmetic and show all of your calculations.)

2. Let $L_k(x)$, $k=0,1,2,\dots,n$ denote the fundamental Lagrange polynomials relative to the n+1 distinct points $x_0, x_1, \dots, x_n$. Use your knowledge of polynomial interpolation to prove Cauchy's equality:

$$\sum_{k=0}^{n} L_k(x) = 1 .$$

3. Use the definition to determine the values of a,b,c and d so that

$$s(x) = \begin{cases} 3 + x - 9x^3 &, \quad 0 \le x \le 1 \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 &, \quad 1 \le x \le 2 \end{cases}$$

is a natural cubic spline on the knot set 0, 1 and 2.

4. Let $x \in R_n$ be the unique solution of the n by n linear system of equations

$$x = Bx + c .$$

Given $x_0 \in R_n$, suppose we generate a sequence of vectors $x_k \in R_n$ by the Jacobi iteration scheme

$$x_{k+1} = Bx_k + c \quad , \quad k = 0,1,2,\ldots$$

Prove: If B is similar to a diagonal matrix D with spectral radius $\rho(D)<1$, then $B^m \to 0$ as $m \to \infty$, and for any initial guess x_0 we have

$$\lim_{k \to \infty} x_k = x \quad .$$

5. Consider the explicit finite difference method for the numerical solution of the normalized heat equation $u_t = u_{xx}$ and for notational convenience let $\lambda = k/h^2$, where the spacial and time steps are respectively $h = \Delta x$ and $k = \Delta t$.
   (a) Show that the local truncation error for the explicit method is in general

$$\frac{k}{2} u_{tt} - \frac{h^2}{12} u_{xxxx} + O(k^2) + O(h^4) \ ,$$

   i.e. of order $O(k) + O(h^2)$.
   (b) Show that the local truncation error is of order $O(k^2) + O(h^4)$ for the special choice $\lambda = 1/6$.

6. Assume r is a root of $x = g(x)$, that $g(x)$ is 3 times continuously differentiable for all x in some neighborhood of r, and that

$$g^{(1)}(r) = g^{(2)}(r) = 0 \quad , \quad g^{(3)}(r) \neq 0 \quad .$$

Prove: If the initial guess x_0 is sufficiently close to r, then the iteration

$$x_{k+1} = g(x_k) \quad , \quad k = 1,2,\ldots$$

will have order of convergence 3.

7. For any nonsingular n by n matrix A, recall that $K(A) = \|A\|\|A^{-1}\|$ is the condition number of A relative to a given subordinate (operator) matrix norm $\|\cdot\|$.
   (a) Establish the following three properties:
   $K(A) \geq 1$,
   $K(\alpha A) = K(A)$ for any nonzero scalar α ,
   and
   $K(AB) \leq K(A)K(B)$ for any nonsingular matrix B .
   (b) Show that for any n by n singular matrix B,
   $K(A) \geq \|A\| / \|A - B\|$.

8. Show that $p(x) = x^2 + 1/8$ is the minimax cubic (or less) degree approximation to the function $|x|$, on the interval $[-1,1]$.