Prelim Exam: Probability, September 1995

(Solve any six problems completely.)

1. (a) State (without proof) Borel-Cantelli lemma (both direct and converse parts; include a precise definition of "infinitely often").
   (b) $\{X_n\}$ are i.i.d. Prove that $E(|X_1|) < \infty$ iff $P\{|X_n| > n \text{ i.o. }\} = 0$.

2. (a) State (without proof) the Levy continuity theorem regarding a sequence of characteristic functions. Apply it to inverstigate the convergence in distribution of $F_n(x)$ where

$$F_n(x) = \begin{cases} 0 & \text{if } x < -n \\ \dfrac{x+n}{2n} & \text{if } -n \le x < n \\ 1 & \text{if } x \ge n. \end{cases}$$

   (b) $\{X_n\}$ are i.i.d. with distribution $F(x)$. Let $S_n = \sum_{k=1}^n X_k$ and F have finite mean μ and finite variance σ^2. Prove that

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to N(0,1), \text{ in distribution as } n \to \infty.$$

3. (a) Define (i) a martingale, (ii) a submartingale.
   (b) Let $\{X_n, n \ge 1\}$ be a sequence of independent r.v.'s. Define $S_n = \sum_{k=1}^n X_k$ and $T_n = \Pi_{k=1}^n X_k$. Prove that
   (i) if $EX_n = 0$ for all n, then $\{S_n\}$ is a martingale.
   (ii) if $EX_n = 1$ for all n, then $\{T_n\}$ is a martingale.
   (c) Let $\{X_n\}$ be a martingale and g be a convex function of R. Prove that $\{g(X_n)\}$ is a submartingale provided $E|g(X_n)| < \infty$, for all $n \ge 1$.

4. Let $\{X_k\}$ be independent r.v.'s such that X_k is uniformly distributed on $[-\frac{1}{k}, \frac{1}{k}]$. Prove that or disprove that the sum $S_n = \sum_{k=1}^n X_k$ convergence almost surely.

5. Two vectors $\{a_i\}_1^n$, $\{b_i\}_1^n$ are orthogonal if $\sum_{i=1}^n a_i b_i = 0$. Let $\{X_i\}$ be independent Gaussian r.v.s such that $\sigma^2(X_i) = \sigma^2(X_j)$ for all i, j. Show that the r.v.s $\sum_{i=1}^n a_i X_i$ and $\sum_{i=1}^n b_i X_i$ are independent provided that $\{a_i\}_1^n$ and $\{b_i\}_1^n$ are orthogonal.

6. (a) State Kolmogorov's strong LLN for i.i.d. r.v.s.
   (b) Let $\{X_n\}$ be exponential i.i.d. r.v.s with parameter λ. Let Z_n denote the number of quantities $X_1, \ldots, X_n$ which exceed 5. Determine the limiting behavior of Z_n/n as $n \to \infty$.

7. X and Y are independent r.v.s. $f : R^2 \to R$ is a Borel measurable function such that the r.v. $f(X,Y)$ has finite mean. $\sigma(X) = \sigma$-algebra generated by X. Show that

$$E[f(X,Y)|\sigma(X)] = \int_R f(X,y) F_Y(dy),$$

   where F_Y is the distribution function of Y.

8. $\{X_n\}$ are i.i.d. r.v.s with $EX_1^2 < \infty$. Prove the CLT for the r.v.s $Y_n = X_1 + 2X_2 + \cdots + nX_n$ as $n \to \infty$.