PhD Preliminary Examination
Numerical Analysis
August 2000

Instructions:. ***Work any ten and only ten of the following eleven problems.*** *Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.* ***Please clearly strike out on this examination the problem that you wish to delete and then hand in both this marked examination and your solution pages.***

1. Assume that f(x) $\in C^3[-h,h]$, where h>0. Let $p_2(x)$ be the polynomial of degree ≤ 2 that interpolates to f(x) at the three distinct points $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. Show that

$$\max_{-h\leq x\leq h} |f(x) - p_2(x)| \leq \frac{\sqrt{3}}{27} h^3 \max_{-h\leq x\leq h} |f^{(3)}(x)|$$

2. Consider the numerical solution of the system of two nonlinear equations

$$f(x,y) = 0$$
$$g(x,y) = 0 \quad .$$

Use Taylor's theorem for a function of two variables to carefully derive Newton's method

$$X_{n+1} = X_n - J_n^{-1} F_n \quad , \quad n = 0, 1, 2, ...$$

for the iterative numerical solution of above system of equations. (Here, $X_n = (x_n , y_n)^T$, $F_n = (f(x_n , y_n), g(x_n \ , y_n))^T$ and J_n is the 2 by 2 Jacobian matrix evaluated at X_n.)

3. Let s(x) be a spline of degree ≤ 1 that interpolates to the function f(x) $\in C^2[x_1, x_2]$ at the n equally space knots $x_1 < x_2 < ... < x_n$. Derive a formula for determining the total number of knots n needed to ensure that the (global) absolute error satisfies

$$|f(x) - s(x)| \leq \varepsilon \quad , \quad x_1 \leq x \leq x_n$$

where $\varepsilon > 0$.

4. Define what is meant by the condition number $K_2(A)$ (relative to the 2-subordinate-matrix norm) of an n by n matrix A. Then compute $K_2(A)$ for the particular 2 by 2 matrix

$$A = \begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix}$$

5. Consider the linear system of equations Ax = b, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 6 & -2 \\ 4 & -3 & 8 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}.$$

Determine and state an appropriate choice for the interation matrix B, that will guarantee the convergence of Jacobi iteration

$$x_{n+1} = B x_n + c \quad , \quad n = 0, 1, \cdots$$

to the solution vector x for this 3 by 3 system. Explain your logic with care.

6. Let $X = C[0,1]$ be normed by $\|\bullet\|_\infty$. Compute the minimax approximation to $\sqrt{x}$ by a polynomial of degree ≤ 1 . Explain your logic with care.

7. Derive the classical three-term recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \quad , \quad n = 1, 2, \ldots$$

for the Chebyshev polynomials of the first kind.

8. Derive the classical 2nd order family of Runge-Kutta methods by determining the four free parameters a, b, α and β so that the one-step method defined by

$$\begin{aligned} y_{n+1} &= y_n + a\, k_1 + b\, k_2 \\ k_1 &= h\, f(x_n, y_n) \\ k_2 &= h\, f(x_n + \alpha h, y_n + \beta k_1) \end{aligned}$$

has local truncation error $O(h^3)$.

9. (a) Use any means at your disposal to derive the simple 1-point Gauss-Legendre quadrature rule

$$\int_{-1}^{1} f(x)dx \approx 2f(0)$$

(b) If this elementary quadrature rule is applied to

$$Tan^{-1}x = \int_{0}^{x} \frac{dt}{1+t^2}$$

we get a rational approximation for the inverse tangent function of the form

$$Tan^{-1}x \approx \frac{Ax+B}{Cx^2+DX+E}$$

Carry out this process and determine the exact numerical values of the constants A,B,C,D and E.

10. Derive and state the classical (finite difference) explicit method for the numerical solution of the heat equation

$$U_t = c^2 U_{xx} \quad , \quad t \geq 0 \quad , \quad a \leq x \leq b$$

for the temperature distribution U(x,t) of a slender, insulated rod of length b - a, subject to the initial temperature distribution

$$U(x,0) = f(x) \quad , \quad a \leq x \leq b$$

and the endpoint boundary conditions

$$U(a,t) = g(t)$$
$$U(b,t) = h(t) \quad , \quad t \geq 0.$$

Be sure to carefully introduce any notation you employ and clearly label any figures you use to help explain your work.

11. Consider the family of n-point Gauss-Legendre quadrature rules

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} w_k f(x_k) + R_n(f)$$

where $R_n(f)$ denotes the quadrature error functional. For any n, it is well known that

(i) $R_n(p) = 0$ for any polynomial p of degree $\leq 2n-1$

(ii) $w_k > 0$, $k = 1, 2, \cdots, n$

(iii) $x_k \in (-1,1)$, $k = 1, 2, \cdots, n$

Assume all of this is given. Suppose f(x) $\in C[-1,1]$. Use the Weirstrass approximation theorem to prove that $R_n(f) \to 0 \; as \; n \to \infty$, i.e. Gauss-Legendre quadrature converges for any continuous function.