Topology Prelim **March, 1997**

1. Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

2. Let X and Y be topological spaces. **Do one of the following.**

   (a) Prove if X and Y are path connected then $X \times Y$ is path connected.

   (b) Prove if X and Y are compact then $X \times Y$ is compact.

3. Let A be a closed subspace of the regular Hausdorff space X and $X \xrightarrow{p} X/\!\sim$ be the natural projection where $\sim$ is the equivalence relation defined by $a \sim b$ if a and b are elements of A. Prove $X/\!\sim$ is Hausdorff if the topology on $X/\!\sim$ is the quotient topology induced from p.

4. Classify all covering spaces of $P \times P$ where P is a 2-dimensional real projective space.

5. Let X be homeomorphic to a 2-dimensional sphere, Y be homeomorphic to a 2-dimensional torus, and Z be the one point union of X and Y.

   (a) Compute the fundamental group of Z.

   (b) Compute $H_*(Z, \mathbb{Z})$.

6. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and f the induced map on the 2-dimensional torus T making the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \downarrow p & & \downarrow p \\ T & \xrightarrow{f} & T \end{array}$$

   commute, where $p(x, y) = (e^{2\pi i x}, e^{2\pi i y})$ is the natural universal covering map of $T = S^1 \times S^1$.

   (a) Prove f is a homeomorphism.

   (b) Prove or disprove: f has a fixed point.

7. Prove there does not exist a retraction of the 3-dimensional sphere S^3 onto a subspace that is homeomorphic to a closed connected 2-manifold.

8. Prove there does not exist a continuous map $S^n \xrightarrow{f} S^{n-1}$ such that $f(-x) = -f(x)$ for all $x \in S^n$ where $S^n \subseteq \mathbb{R}^{n+1}$ is the unit sphere and $n \geq 1$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX