SYLLABUS FOR MATH 4000
Modern Algebra and Geometry I

This is the first of a two-course sequence in abstract algebra. MATH 4000 covers the basics of ring and field theory (generally proceeding from the concrete to the abstract). MATH 4010 covers group theory, group actions, Galois theory, and—according to the instructor's taste—topics in projective geometry.

Text: T. Shifrin, *Abstract Algebra: A Geometric Approach*, Prentice Hall, 1996. See errata at `http://www.math.uga.edu/~shifrin/Typos.html`.

Chapter 1 The Integers: Induction, binomial theorem; Euclidean algorithm, prime numbers and factorization; modular arithmetic, Chinese remainder theorem; $\mathbb{Z}_m$, rings, integral domains, and fields. (3.5 weeks)

Do some examples in class of proofs using modular arithmetic (e.g., can 163 be the sum of two squares? what are the last two digits of 7^{486}). De-emphasize the formula given in Theorem 3.7; rather, expect the students to derive the solution from first principles. In the interest of time, you may want to omit Theorem 3.8 entirely.

Chapter 2 From the Integers to the Complex Numbers: Rational and real numbers, complex numbers, quadratic and cubic formulas; the isometries of $\mathbb{R}$ and $\mathbb{C}$. (3 weeks)

Do §1 somewhat carefully, but give short shrift to §2. Similarly, plan to derive the quadratic formula and devote at most a day to the cubic formula.

Chapter 3 and Chapter 5, §1 Polynomials, vector spaces, and dimension: Euclidean algorithm, roots of polynomials, irreducibility of polynomials with integer coefficients. Vector spaces, dimension, applications to degree of field extensions and splitting fields. (4 weeks)

It is probably pedagogically smoother to reorganize things somewhat. After completing §1, introduce $F[\alpha]$ and use the Euclidean algorithm to show it's a field (including concrete examples of finding $1/\alpha$). Then move on to discuss irreducibility in $\mathbb{Q}[x]$ and cover §3. At this point, motivated by the definition of splitting field, move on to a linear algebra review and §5.1, with the concomitant dimension theory and the main result, Lemma 1.6 of Chapter 5. From the multiplicativity result, Prop. 1.5, and techniques such as in Example 5, we can now compute degrees of various field extensions and finish up §2 of Chapter 3.

It is a good idea to emphasize the analogies between $\mathbb{Z}$ and $F[x]$, between finding a multiplicative inverse in $\mathbb{Z}_m$ and that in $\mathbb{Q}[\alpha]$. (It works! When you get to Chapter 4, most of the students will grasp computations in quotient rings.) For the future teachers, issues like factoring to find roots (and when this works!) are important (along with different proofs that if $f(x) \in F[x]$ has degree n, then $f(x)$ has *at most* n roots in F).

Chapter 4 Homomorphisms and Quotient Rings: Ring homomorphisms, ideals, fundamental homomorphism theorem and isomorphisms; Gaussian integers (optional topic). (3 weeks)

It is a good idea to include some more concrete examples of the Fundamental Homomorphism Theorem both in lecture and in exercises. For example, prove that $\phi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_{12}$, $\phi(x, y) = 4x-3y \pmod{12}$, is a homomorphism, and deduce that $\mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_{12}$. For the students who will go on to study number theory, it is helpful to understand that the full strength of the Chinese Remainder Theorem is a ring isomorphism. If you have time to do the section on Gaussian integers, you can do comparable applications there.

Chapter 5, §2 Field Extensions: compass and straightedge constructions. (1 week)