MATHEMATICS COMPETITION

University of Georgia

February 2, 2002

Problem 1. What is the sum of all the divisors of 36, including 1 and itself?

A. 50
B. 55
C. 73
D. 90
E. 91

Problem 2. A cube is inscribed in a ball. What is the ratio of the volume of the ball to the volume of the cube?

A. $\pi/6$
B. $\pi/3$
C. $\sqrt{3}\pi/2$
D. $4\pi/3$
E. None of the above

Problem 3. Find

$$\frac{1}{3+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{3}+1}\,.$$

A. $1/2$
B. 1
C. 2
D. $\sqrt{5}$
E. 4

Problem 4. There are three distinct lines and two distinct circles. What is the largest number of points in which at least two of these figures intersect?

A. 14
B. 15
C. 16
D. 17
E. 18

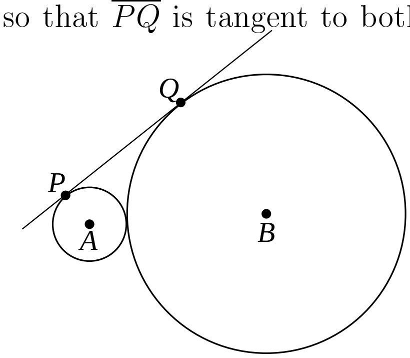
Problem 5. Find the number of triples (a, b, c) of positive integers with $a \geq b \geq c$ that satisfy the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1\,.$$

A. 1
B. 2
C. 3
D. 4
E. more than 4

Problem 6. Circle A has radius 1, circle B has radius 4. Points P and Q are chosen on circles A and B, respectively, so that $\overline{PQ}$ is tangent to both circles. Find PQ.

A. $2\sqrt{2}$
B. 3
C. 4
D. $\sqrt{17}$
E. $14/3$



Problem 7. On January 1, 2002, three plants stood in a line on Sophie's windowsill, from left to right: a rose, violet and tulip. Each morning, when Sophie waters the plants, she changes the plant on the left with the one in the center. Each afternoon, her younger sister waters the plants, and changes the plant on the right with the one in the center. What will be the order of the plants on January 1, 2003, 365 days later?

A. rose, tulip, violet
B. violet, rose, tulip
C. violet, tulip, rose
D. tulip, violet, rose
E. tulip, rose, violet

Problem 8. Point P lies within parallelogram $ABCD$. If $\triangle ABP$ has area 5, $\triangle BCP$ has area 6, and $\triangle APD$ has area 8, then find the area of $\triangle CDP$.

A. 3
B. 7
C. 8
D. 9
E. None of the above

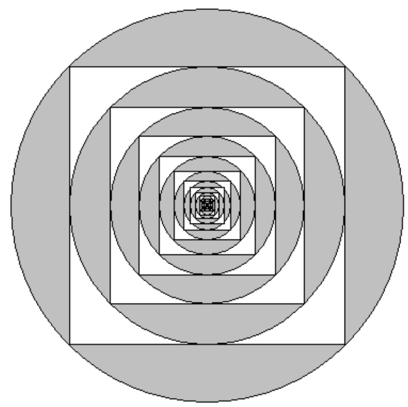
Problem 9. Valery multiplied all integers from 1 through his age and got

$$87,178,291,200\,.$$

How old is Valery?

A. 11
B. 12
C. 13
D. 14
E. 15

Problem 10. What is the ratio of the area of the unshaded region below to the area of the largest circle?



A. $2-\dfrac{4}{\pi}$
B. $\dfrac{4-\pi}{2\pi-4}$
C. $\dfrac{\pi-2}{4}$
D. $2\pi-6$
E. $\dfrac{4}{\pi}-1$

Problem 11. What real number x satisfies the (infinite) equation

$$e^{x e^{x e^{x^{\cdot^{\cdot^{\cdot}}}}}} = 2\,?$$

A. $(\ln 2)/2$
B. $\ln 2$
C. $\sqrt{\ln 2}$
D. $\sqrt{2}/2$
E. None of the above

Problem 12. A fair die was thrown three times. What is the probability that each result was greater than the one before it?

A. 1/12
B. 5/54
C. 1/9
D. 1/6
E. 1/4

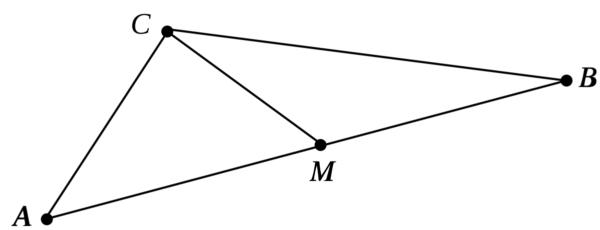
Problem 13. A square is divided by parallel lines into 25 rectangles, as indicated in the diagram below (not drawn to scale). Areas of some of them are provided. If the area of the rectangle marked with the "?" sign is p/q as a fraction in lowest terms, what is $p+q$?

A. 17
B. 54
C. 131
D. 211
E. None of the above

?				5
			4	9
		3	8	
	2	7		
1	6			

Problem 14. Given $\triangle ABC$ with $AB = 4$, $BC = 3$, and $AC = 2$, let M be the midpoint of $\overline{AB}$. Then determine CM.

A. $\sqrt{10}/2$
B. $\sqrt{3}$
C. $5/2$
D. $2\sqrt{2}$
E. 3



Problem 15. What is the smallest number of digits one can write in a row so that the following is true: One can obtain any 3-digit number from this sequence by striking out some of the digits and concatenating the rest?

A. 29
B. 45
C. 901
D. 1000
E. None of the above

Problem 16. A radio-controlled car starts from some point. It moves in a certain direction until a command is given, at which time it turns 17 degrees left or right. What is the smallest number of commands required to return the toy to the starting point?

A. 10
B. 11
C. 21
D. 22
E. None of the above

Problem 17. If $f(x,y) = x^2 - xy + y + 1$, what is the sum of the coefficients of the terms of $f(x,y)^3$ in which x appears to an odd power?

A. -8
B. -27
C. -28
D. -36
E. -64

Problem 18. How many pairs (x,y) of positive integers satisfy the relation

$$1! + 2! + \cdots + x! = y^2\,?$$

A. 1
B. 2
C. 3
D. more than 3, but finitely many
E. infinitely many

Problem 19. Suppose $ab = 4$ and $a + b = 3$. Find $1/a^3 + 1/b^3$.

A. $-3/32$
B. $1/8$
C. $-1/8$
D. $9/64$
E. $-9/64$

Problem 20. Find the remainder when 11! is divided by 13.

A. 1
B. 12
C. 6
D. 7
E. None of the above

Problem 21. A computer printed out the (base 10) numerals 2^{2002} and 5^{2002}. How many digits did it print out?

A. 2000
B. 2001
C. 2002
D. 2003
E. 2004

Problem 22. What is the greatest number of regions into which four planes can divide three-dimensional space?

A. 8
B. 11
C. 14
D. 15
E. 16

Problem 23. In a certain base (different from 10) one has

$$447^2 = 172,501\,.$$

What is 17^2 in this base?

A. 201
B. 261
C. 281
D. 289
E. 341

Problem 24. What is

$$\cos\frac{\pi}{9}\cos\frac{4\pi}{9}\cos\frac{7\pi}{9}\,?$$

A. $-1/8$
B. $-\sin(\pi/9)$
C. $-1/4$
D. $-\pi/18$
E. None of the above

Problem 25. Exactly one of the following numbers, written in base 6, is prime. Which is it?

A. 121254525
B. 140320254
C. 302021431
D. 305313415
E. 353042123