Real Analysis Qualifying Examination

Spring 2023

The five problems on this exam have equal weighting. To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. (a) Demonstrate the existence of a positive function f that is both integrable and continuous on $\mathbb{R}$, but has the property that $\limsup_{x\to\infty} f(x) = \infty$.

   (b) Prove that if f is both integrable and uniformly continuous on $\mathbb{R}$, then $\lim_{|x|\to\infty} f(x) = 0$.

2. (a) Let $G \subseteq \mathbb{R}\times\mathbb{R}$ be open and $f : G \to \mathbb{R}$ be continuous. Prove that

$$F(x) := \sup_{\{y\,:\,(x,y)\in G\}} f(x,y)$$

   defines a Borel measurable function F on $\mathbb{R}$.
   Hint: Recall that F is Borel measurable if $F^{-1}((a,\infty])$ is a Borel set for all $a\in\mathbb{R}$.

   (b) Prove that if g is a continuous function on $\mathbb{R}$, then the set of points where g is differentiable is a Borel measurable set, and that on this set g' is a Borel measurable function.
   Hint: For each $n\in\mathbb{N}$ consider the functions

$$f_n(x,y) = \frac{g(x+y)-g(x)}{y} \text{ restricted to the open sets } G_n = \{(x,y)\,:\, x\in\mathbb{R} \text{ and } 0<|y|<1/n\}.$$

3. (a) Let $E\subseteq[0,1]$ be measurable with $m(E)=0$. Prove that

$$m(\{y\in[0,1]\,:\,y^2\in E\}) = 0.$$

   Hint: First consider when $E\subseteq[a,1]$ for some $a>0$.

   (b) Prove that if f is a non-negative measurable function on $[0,1]$, then

$$\int_{[0,1]} f(x)\,dx = \int_{[0,1]} f(y^2)\,2y\,dy.$$

   Hint: First consider when f is the characteristic function of an open interval, then the characteristic function of an open set,..., and ultimately when f is a simple function.

4. Let

$$F(t) = \int_0^\infty e^{-x^3\sin(t)}\,dx.$$

   (a) Prove that F is a well-defined real-valued differentiable function for all $t\in(0,\pi)$ with derivative

$$F'(t) = -\cos(t)\int_0^\infty x^3 e^{-x^3\sin(t)}\,dx.$$

   (b) Prove that F has the further property that

$$\lim_{t\to0^+} F(t) = \lim_{t\to\pi^-} F(t) = \infty.$$

5. (a) Show, without appealing to methods from complex analysis, that

$$A := \int_0^\infty \frac{1}{(1+y)\,y^{1/2}}\,dy < \infty.$$

   (b) Let $h(x) = \log(x)/x$ for all $x\in[0,\infty)$. Prove that $h\in L^2([0,\infty))$ with $\|h\|_{L^2([0,\infty))}\le A$, where A is the constant from part (a) above, by showing that

$$\left|\int_0^\infty \frac{\log(1+x)}{x} f(x)\,dx\right| \le A\,\|f\|_{L^2([0,\infty))}$$

   for all $f\in L^2([0,\infty))$.
   Hint: Use the fact that $\log(1+x) = \displaystyle\int_0^x \frac{1}{1+y}\,dy$.